Univariate Polynomial Solutions of Algebraic Difference Equations

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Abstract

Contrary to linear difference equations, there is no general theory of difference equations of the form $G(P(x-\tau_1),\ldots,P(x-\tau_s))+G_0(x){=}0$, with $\tau_i \in \mathbb{K}$, $G(x_1,\ldots,x_s) \in \mathbb{K}[x_1,\ldots,x_s]$ of total degree $D \geq 2$ and $G_0(x) \in \mathbb{K}[x]$, where $\mathbb{K}$ is a field of characteristic zero. This article is concerned with the following problem: given τ_i, G and G_0, find an upper bound on the degree d of a polynomial solution $P(x)$, if it exists. In the presented approach the problem is reduced to constructing a univariate polynomial for which d is a root. The authors formulate a sufficient condition under which such a polynomial exists. Using this condition, they give an effective bound on d, for instance, for all difference equations of the form $G\big(P(x-a),P(x-a-1),P(x-a-2)\big)+G_0(x)=0$ with quadratic G, and all difference equations of the form $G\big(P(x),P(x-\tau)\big)+G_0(x)=0$ with G having an arbitrary degree.

Key words: difference equation, elementary symmetric polynomials, power-sum symmetric polynomials, Newton-Girard identities, system of linear equations.

1. Introduction

This article considers polynomial solutions of difference equations of the form

$$G(P(x-\tau_1),\ldots,P(x-\tau_s))+G_0(x)=0 \tag{1}$$

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where $G(x_1,\ldots,x_s) \in \mathbb{K}[x_1,\ldots,x_s]$ is a polynomial of *total degree* $D \geq 2$ in s variables, $\mathbb{K}$ is a field of characteristic zero, $G_0(x) \in \mathbb{K}[x]$ and $\tau_i \in \mathbb{K}$ are pairwise different and ordered so that $\tau_1 < \ldots < \tau_s$. The aim is to find a bound on the degree d of polynomial solutions $P(x)$ if such solutions and a bound exist.

It is worth to note that there are difference equations which are solvable by a polynomial of any degree (therefore, no bound exists), e.g.:

$$\begin{aligned} &P(x)P(x-2)P(x-3) - 2P(x-1)^2P(x-3) + P(x-1)P(x-2)^2 + \\ &P(x)P(x-1)P(x-3) - 2P(x)P(x-2)^2 + P(x-1)^2P(x-2) = 0. \end{aligned} \tag{2}$$

It is solved by any factorial power $g_n(x) = (x+a)(x+a-1)\ldots\big(x+a-(n-1)\big)$. The proof resembles the technique for differential equations from the article (van den Essen, 1992) and can be found in Section 4. Moreover, the statement can be checked by direct substitution using a computer algebra system.

In the present article the equations of form (1) are called *algebraic difference equations with constant coefficients*. The terminology "with constant coefficients" is used because one considers polynomials $G(x_1,\ldots,x_s)$ with coefficients which are independent of x. The authors believe that extending the proposed method to difference equations where the coefficients of $x_1^{i_1}\cdots x_s^{i_s}$ depend on x will require only some technical adjustments. However it is left to future work because the results even for constant coefficients require technically involved computations.

Notation

The present article involves reasoning about symbolic vectors, products of powers and indexed polynomials whose coefficients are polynomials as well. Therefore technical overhead in formal reasoning is inevitable. The following list of the most frequently used notation, which can be used as a general reference, should help to handle this overhead:

notation	denoted object
$\mathbf{r}$	the vector $(\rho_1,\ldots,\rho_d) \in \overline{\mathbb{K}}^d$ of the roots of $P(x) \in \mathbb{K}[x]$,
	where $\overline{\mathbb{K}}$ is the algebraic closure of $\mathbb{K}$
T	the set $\{(\tau_{k_1},\ldots,\tau_{k_D}) \mid 1 \leq k_1 \leq \ldots \leq k_D \leq s\}$ of all the ordered vectors
	with entries from the ordered set $\{\tau_1,\ldots,\tau_s\}$
$\mathbf{t}$	a vector $(t_1,\ldots,t_D)$ that ranges over T
$\mathbf{u}_\ell$ and $\mathbf{v}_\ell$	vectors $(u_1,\ldots,u_\ell)$ and $(v_1,\ldots,v_\ell)$ respectively
$\mathbf{i}_\ell$ and $\mathbf{j}_\ell$	vectors $(i_1,\ldots,i_\ell)$ and $(j_1,\ldots,j_\ell)$ respectively
$\mathbf{v}_\ell^{\mathbf{i}_\ell}$ and $\mathbf{u}_\ell^{\mathbf{j}_\ell}$	monomials $v_1^{i_1}\cdots v_\ell^{i_\ell}$ and $u_1^{j_1}\cdots u_\ell^{j_\ell}$ respectively
$p_\ell(y_1,\ldots,y_m)$	the power-sum symmetric polynomial $y_1^\ell + \ldots + y_m^\ell$
$\mathbf{p}_\ell(y_1,\ldots,y_m)$	the vector $\big(p_1(y_1,\ldots,y_m)\ldots,p_\ell(y_1,\ldots,y_m)\big)$
$\mathbf{0}_\ell$	the ℓ-dimensional null-vector $(0,\ldots,0)$

The computations supporting the presented results are mainly computer-aided. This means that reading some formulæ might not be easy. Moreover, this explains why such results could not appear a few decades ago or earlier: the field of computer algebra was not developed enough.

The approach in a nutshell and the outline of the paper

Let d denote the degree of a solution $P(x) \in \mathbb{K}[x]$ of equation (1). Our aim is to construct a *degree polynomial* for equation (1), that is a univariate polynomial for which d is a root. Degree polynomials for linear recurrence relations with polynomial coefficients are defined, e.g., in the book (Petkovšek et al., 1996).

The approach presented in this article is based on equating the corresponding coefficients in the right- and left-hand side of an identity between two polynomials. This approach is applied not to equation (1), but to the equivalent equation (3) below:

$$G_D\big(P(x-\tau_1),\dots,P(x-\tau_s)\big) = -G_{<D}\big(P(x-\tau_1),\dots,P(x-\tau_s)\big) - G_0(x) \qquad (3)$$

where $G(x_1,\dots,x_s)$ is represented as the sum $G_D(x_1,\dots,x_s) + G_{<D}(x_1,\dots,x_s)$ with G_D being the homogeneous part with total degree D and $G_{<D}$ containing the terms of G with total degrees $< D$. [1]

Without loss of generality one can assume that $d(D-1) > deg(G_0)$, otherwise clearly we have a bound $d \le deg(G_0)/(D-1)$. Then the degree w.r.t. x on the right-hand side of equation (3) is at most $d(D-1)$. The degree w.r.t. x of the left-hand side is at most dD. All coefficients of $x^{dD}, x^{dD-1}, \dots, x^{d(D-1)+1}$ on the left-hand side must vanish because $dD > d(D-1)$. In Section 2 we give a necessary set-up and show that these coefficients can be expressed in terms of the *power-sum symmetric polynomials* evaluated at the roots $\mathbf{r}$ of $P(x)$. Note that for $P(x) \in \mathbb{K}[x]$ the values $p_\ell(\mathbf{r})$ are in $\mathbb{K}$ even if there are roots in $\overline{\mathbb{K}} \setminus \mathbb{K}$. One constructs polynomials $S_\ell(u_0, u_1, \dots, u_\ell)$ such that the coefficient of $x^{dD-\ell}$ on the l.h.s. of equation (3) is equal to $S_\ell\big(d, p_1(\mathbf{r}), \dots, p_\ell(\mathbf{r})\big)$. In general, S_ℓ cannot be taken as degree polynomials, because they depend on $\ell + 1$ variables.

In this article we analyse some cases when the variables $u_1, \dots, u_\ell$ can be eliminated from a certain equation $S_\ell(u_0, u_1, \dots, u_\ell) = 0$, so that the degree polynomial $Q_0(u_0)$ is equal to $S_\ell(u_0, \mathbf{0}_\ell)$. The *framework lemma* in Section 2 gives a sufficient condition for such elimination to be possible. In Sections 3 and 5, respectively, we consider two independent cases for which the conditions of the framework lemma hold and therefore the degree d can be bounded:

- let $\mathcal{L}$ denote the set $\{\ell | S_\ell(u_0, \mathbf{0}_\ell)$ is not everywhere zero$\}$; if $\mathcal{L} \neq \emptyset$ and $L := \min(\mathcal{L}) \le 5$ then either $d \le \max\{L, \deg(G_0)/(D-1)\}$, or d is a root of $S_L(u_0, \mathbf{0}_L)$, (see Theorem 4 and the example in Section 6),
- $d \le \max\{D, \deg(G_0)/(D-1)\}$ for all difference equations of the form $G\big(P(x), P(x-\tau)\big) + G_0(x) = 0$ (see Theorem 6).

In Section 7 we sum up the results and outline future work. Technical details of the proofs can be found in the Appendix. The proofs are supported by calculations in Maple (download `nonlindifeq.tar.gz`, available on the site `http://resourceanalysis.cs.ru.nl` under the item *Technical reports*).

[1] Subsequently, we say that *a monomial* in the variables $x_1, \dots, x_n$ is the product of powers of x_i. It has the form $x_1^{i_1} \dots x_n^{i_n}$. A *term* is a product of powers multiplied by a constant.

Related work

The bound $d \leq D$ for $G(P(x), P(x-\tau)) = 0$ with vanishing $G_0(x)$ resembles the result $d = D$ for ordinary difference equations of the form $G(P(x), P(x-1)) = 0$ where the polynomial $G(x_1, x_2)$ is irreducible in rational field extension and D is the total degree of G, see (Feng et al., 2008). The latter gives the *precise* degree of a polynomial solution for an irreducible polynomial G whereas we give just an upper bound. However, we do not demand irreducibility of G. Since G is the product of its irreducible factors, applying the result of the article (Feng et al., 2008) for each of them gives $d \leq D$.

In the article (Tang et al., 2010) the authors investigate the global behavior of solutions of non-linear difference equations of the form $x_{n+1} = (\alpha + x_n)/(A + Bx_n + x_{n-k})$, where $n \geq 0$, the parameters are positive real numbers and the initial conditions $x_{-k}, \ldots, x_0$ are non-negative real numbers, $k \geq 2$. One of the results is that every solution is bounded from above and from below by positive constants. In (Öcalan, 2009) one gives necessary and sufficient conditions for the oscillation of solutions x_n of nonlinear difference equations of the form $x_{n+1} - x_n + \sum_{i=1}^{m} p_i f_i(x_{n-k_i}) = 0$ where $k_i \in \{\ldots, -2, -1\}$ and $p_i < 0$ for $1 \leq i \leq m$. Moreover, the result is generalized to equations with non-constant coefficients, p_{in}.

A bound on the degree of polynomial solutions of linear homogeneous recurrence relations with polynomial coefficients $P(n) = G(n, P(n-1), \ldots, P(n-s))$ is obtained in the article (Abramov, 1989). It is done via a degree polynomial. In the article (Mezzarobba and Salvy, 2010) a similar problem is considered for complex polynomials, satisfying linear recurrence relations with rational-polynomial coefficients. The authors constructively define a real sequence that dominates the absolute value of the complex polynomial sequence. In (Borcea et al., 2011) one gives the asymptotic ratio $\lim_{n\to\infty} \frac{f_{n+1}(\mathbf{x})}{f_n(\mathbf{x})}$ for $f_n(\mathbf{x})$ satisfying a linear recurrence equation of the form $f_{n+k}(\mathbf{x}) + \sum_{i=1}^{k} \phi_{i,n}(\mathbf{x}) f_{n+k-i}(\mathbf{x}) = 0$, with $n \geq k-1$.

In the article (Rolanía and Lagomasino, 2007) the authors consider asymptotic behavior of recurrence relations of the form $f(n)(z) = b(n)(z) f(n-1) + a^2(n)(z) f(n-2)(z)$, where $b(n)(z)$, $a(n)(z)$ are analytic in a certain complex domain.

Orthogonal polynomials on the real line always satisfy a three-term recurrence relation (see e.g. (van Assche and Foupouagnagni, 2003)):

$$P(n+1)(x) = (x - \beta(n))P(n)(x) - \gamma(n)P(n-1)(x)$$

It is still linear w.r.t. $P(n)(x)$ (but the polynomial solution is two-variate). However, the recurrence relations for the coefficients $\beta(n)$ and $\gamma(n)$ satisfy *non-linear recurrence relations* of a particular form or systems of such relations. An example of such system may be found in the article (van Assche and Foupouagnagni, 2003). It defines the coefficients $\beta(n)$ and $\gamma(n)$ such that $P(n)(x)$ is a generalized Charlier polynomial. The asymptotic behavior is such coefficients is studied as well.

In (Máté and Nevai, 1985) the authors study asymptotics for the recurrence relations of the form $H(f(n), f(n+1), \ldots, f(n+s), 1/n) = 0$, where H is a complex-valued function of $s+2$ real variables all of whose partial derivatives of order $\leq m$ are continuous in a neighborhood of the origin $\bar{0}$ and $\sum_{j=0}^{s} z^j \frac{\partial H}{\partial x_j}(\bar{0}) \neq 0$ for all complex numbers z with $|z| = 1$. The authors define numbers $c_1, \ldots, c_m$ such that $f(n) = \sum_{l=1}^{m} c_l n^{-l} + o(n^{-m})$. In the later publications the authors extend this result to *systems* of such recurrence relations. In our opinion this result cannot be applied to solve our problem for $\tau_i = i$ and $P(x) = G\big(P(x-1), \ldots, P(x-s), x\big)$ because we cannot transform this equation into the equivalent one of the form above. At first sight, one could analyse the equation $P(1/n) = G\big(P(1/n-1), \ldots, P(1/n-s), 1/n\big)$ obtained as a partial case of the original equation for $x = 1/n$. However, the expected substitution $f(n) = P(1/n)$ to obtain an equation over f is impossible due to $P(1/n-i) \neq P\big(1/(n-i)\big) = f(n-i)$.

2. Coefficients of x in $G_D\big(P(x-\tau_1), \ldots, P(x-\tau_s)\big)$ as symmetric polynomials

2.1. Polynomial difference equations $G\big(P(x-\tau_1), \ldots, P(x-\tau_s)\big) = 0$

Before studying difference relations in detail, note that a *recurrence relation* with a polynomial solution defined on natural numbers determines a difference equation with the same schema.

Lemma 1. Let a polynomial $P(x)$ satisfy $G(P(n-1), \ldots, P(n-s)) + G_0(n) = 0$ for all integer $n \geq n_0$, for some n_0. Then $G(P(x-1), \ldots, P(x-s)) + G_0(x)$ for all real $x \in \mathcal{R}$.

Proof. From the condition of the lemma it follows that the polynomial in x, namely $G(P(x-1), \ldots, P(x-s)) + G_0(x)$, is equal to zero in some $\deg(P)+1$ pairwise different points. From this follows that it is zero for all $x \in \mathcal{R}$. $\square$

This property makes the difference equation analysis applicable to analysis of recurrence relations.

Lemma 2. Let a function $f(x)$ (which is not necessarily a polynomial) satisfy $G\big(f(x), f(x-1), \ldots, f(x-s)\big) = 0$ for all $x \in \mathbb{K}$. Then any $g(x)$, such that $g(x) = f(x+a)$ for some $y \in \mathbb{K}$, satisfies the equation $G\big(g(x), g(x-1), \ldots, g(x-s)\big) = 0$ as well.

Proof. Since for all x the identity $G\big(f(x), f(x-1), \ldots, f(x-s)\big) = 0$ holds then for $x := x+a$ one has $G\big(f(x+a), f(x+a-1), \ldots, f(x+a-s)\big) = 0$ as well. That is $G\big(g(x), g(x-1), \ldots, g(x-s)\big) = 0$. $\square$

We consider a multivariate polynomial $G_D(x_1, \ldots, x_s) = \sum_{i_1+\ldots+i_s=D} a_{i_1\ldots i_s} x^{i_1} \cdots x^{i_s}$ of total degree D. Re-index the coefficients $a_{i_1\ldots i_s}$ of $G_D(x_1, \ldots, x_s)$ in such a way that, for instance, the coefficient a_{20} of $x_1^2 = x_1x_1$ becomes $\alpha_{(\tau_1,\tau_1)}$ and the coefficient a_{11} of x_1x_2 becomes $\alpha_{(\tau_1,\tau_2)}$. Consider another example: take $G_5(x_1, x_2, x_3)$ of degree $D = 5$ with $s = 3$. Its term $a_{230}x_1^2x_2^3$ is represented as $\alpha_{(\tau_1,\tau_1,\tau_2,\tau_2,\tau_2)}x_1x_1x_2x_2x_2$. In general, the reindexation $I : \{(i_1 \ldots i_s) \,|\, i_1 + \ldots + i_s = D,\ i_j \in \mathbb{N}\} \to T = \{(\tau_{k_1}, \ldots, \tau_{k_D}) \,|\, 1 \leq k_1 \leq \ldots \leq k_D \leq s\}$ maps $(i_1 \ldots i_s)$ to $\mathbf{t} = (t_1, \ldots, t_D) = (\tau_1^{(i_1)}, \tau_2^{(i_2)}, \ldots, \tau_s^{(i_s)})$, where $\tau^{(i)}$ denotes τ repeated i times. Clearly, I is a bijection since the τ_i are pairwise distinct.

Lemma 3. The re-indexing I that sends $(i_1 \ldots i_s)$ to $T = (\tau_1^{(i_1)}, \ldots, \tau_s^{(i_s)})$, is a bijection.

Proof. By its definition, I is a map (i.e. is functional and everywhere defined). We have to prove that it is injective and surjective.

Injectivity is proven by contradiction. Assume that there are two different indices, $(i_1 \ldots i_s)$ and $(i'_1 \ldots i'_s)$, that are mapped to the same $\mathbf{t} = (\tau_1^{(i_1)}, \ldots, \tau_s^{(i_s)})$. Let $\ell = \min\{j | i_j \neq i'_j\}$. Therefore, $(i'_1 \ldots i'_s) = (i_1 \ldots i_{\ell-1}, i'_\ell, \ldots, i'_s)$.

Now, $(\tau_1^{(i_1)}, \ldots, \tau_\ell^{(i_\ell)}, \ldots, \tau_s^{(i_s)}) \neq (\tau_1^{(i_1)}, \ldots, \tau_{(\ell-1)}^{(i_{\ell-1})}, \tau_\ell^{(i'_\ell)}, \ldots \tau_s^{(i'_s)}) = \mathbf{t}$, which is a contradiction. So, the map I is an injection.

To prove surjectivity, fix any $\mathbf{t} \in T$. It easy to see, that by the definition of T there exist $i_j \in \mathbb{N}$, such that $i_1 + \ldots + i_s = D$ and $\mathbf{t} = (\tau_1^{(i_1)}, \ldots, \tau_s^{(i_s)})$. We take $\mathbf{i} = (i_1, \ldots, i_s)$. Trivially, $I(\mathbf{i}) = (\tau_1^{(i_1)}, \ldots, \tau_s^{(i_s)}) = \mathbf{t}$. Therefore, the map I is a surjection. $\square$

With this reindexation we write

$$G_D(x_1, \ldots, x_s) = \Sigma_{\mathbf{t}=(\tau_{k_1}, \ldots, \tau_{k_D}) \in T} \alpha_{\mathbf{t}} x_{k_1} \ldots x_{k_D}.$$

For instance, for $D = 2, s = 3$ one has

$$T = \{(\tau_1, \tau_1), (\tau_1, \tau_2), (\tau_1, \tau_3), (\tau_2, \tau_2), (\tau_2, \tau_3), (\tau_3, \tau_3)\}$$

and for the polynomial $G_2(x_1, x_2, x_3) = x_1^2 - 2x_1x_2 + x_3^2$ the reindexation yields $\alpha_{(\tau_1, \tau_1)} = 1$, $\alpha_{(\tau_1, \tau_2)} = -2$, $\alpha_{(\tau_3, \tau_3)} = 1$ and $\alpha_{(\tau_1, \tau_3)} = \alpha_{(\tau_2, \tau_2)} = \alpha_{(\tau_2, \tau_3)} = 0$. Consider another example: $G(x_1, \ldots, x_5) = a_{20000}x_1^2 - 2a_{10010}x_1x_4 + a_{00002}x_5^2$. The corresponding re-indexed polynomial is $\alpha_{11}x_1^2 - 2\alpha_{14}x_1x_4 + \alpha_{55}x_5^2$. Here we have $D = 2$, $s = 5$, and all the possible indices are in the set

$$K = \{(1,1), (1,2), (1,3), (1,4), (1,5), (2,2), (2,3), (2,4), (2,5), (3,3), (3,4), (3,5), (4,4), (4,5)\}$$

Here we have that all α_{ij}, except α_{11}, α_{14}, α_{55}, vanish.

Let the polynomial P be represented via its roots: $P(x) = a_d(x - \rho_1) \ldots (x - \rho_d)$. The product $P(x - t_1) \ldots P(x - t_D)$ is equal to $a_d^D \prod_{i=1}^D \prod_{j=1}^d (x - t_i - \rho_j)$. For this product one wants to find the coefficients $\varepsilon_\ell(\mathbf{t}, \mathbf{r})$ of $x^{Dd-\ell}$, where $0 \leq \ell \leq d - 1$. The sums $(t_i + \rho_j)$, where $1 \leq i \leq D$, $1 \leq j \leq d$ are obviously the (only) roots of the polynomial $\prod_{i=1}^D \prod_{j=1}^d (x - t_i - \rho_j)$. Therefore, its coefficients $\varepsilon_\ell(\mathbf{t}, \mathbf{r})$ are represented via *the elementary symmetric polynomials* $e_\ell(y_1, \ldots, y_{dD}) := \sum_{1 \leq i_1 < i_2 \ldots < i_\ell \leq dD} y_{i_1} \cdots y_{i_\ell}$ and $e_0(y_1, \ldots, y_{dD}) := 1$ (Macdonald, 1979) in the standard way:

$$\varepsilon_\ell(\mathbf{t}, \mathbf{r}) = (-1)^\ell e_\ell(t_1 + \rho_1, \ldots, t_i + \rho_j, \ldots, t_D + \rho_d). \tag{4}$$

Lemma 4. If a polynomial P of degree d solves equation (3) and $d > \ell$ for some $\ell \geq 0$ then the roots $\mathbf{r}$ of $P(x)$ must satisfy the identity

$$\textstyle\sum_{\mathbf{t} \in T} \varepsilon_\ell(\mathbf{t}, \mathbf{r}) \alpha_{\mathbf{t}} = 0. \tag{5}$$

Proof. Due to $d > \ell$ one has that $dD - \ell > d(D - 1)$. Since P solves equation (3), the coefficients $a_d^D \sum_{\mathbf{t} \in T} \varepsilon_\ell(\mathbf{t}, \mathbf{r}) \alpha_{\mathbf{t}}$ of $x^{dD-\ell}$ on the l.h.s. of equation (3) must vanish. Having $a_d \neq 0$, one obtains identity (5). $\square$

Lemma 4 does not give direct information about d, since each $\varepsilon_\ell(\mathbf{t},\mathbf{r})$ depends on d implicitly: d is the dimension of $\mathbf{r}$. To obtain an explicit equation for d from equation (5), employ power-sum symmetric polynomials and the Newton-Girard formulæ (Macdonald, 1979):

$$e_\ell(y_1,\dots,y_m) = (1/\ell)\sum_{\kappa=1}^{\ell}(-1)^{\kappa-1}e_{\ell-\kappa}(y_1,\dots,y_m)p_\kappa(y_1,\dots,y_m).$$

One can easily check by the definition of p_κ and the binomial formula, that

$$p_\kappa(t_1+\rho_1,\dots,t_i+\rho_j,\dots,t_D+\rho_d) = \sum_{i=1}^{D}\sum_{j=1}^{d}(t_i+\rho_j)^\kappa = \sum_{\lambda=0}^{\kappa}\binom{\kappa}{\lambda}p_{\kappa-\lambda}(\mathbf{t})p_\lambda(\mathbf{r}). \tag{6}$$

In more detail,

$$\begin{aligned} p_\kappa(\mathbf{t}+\mathbf{r}) = \quad & \textstyle\sum_{i=1}^{D}\sum_{j=1}^{d}(t_i+\rho_j)^\kappa = \sum_{i=1}^{D}\sum_{j=1}^{d}\sum_{\lambda=0}^{\kappa}\binom{\kappa}{\lambda}\rho_j^\lambda t_i^{\kappa-\lambda} = \\ & \textstyle\sum_{\lambda=0}^{\kappa}\binom{\kappa}{\lambda}\sum_{i=1}^{D}\sum_{j=1}^{d}\rho_j^\lambda t_i^{\kappa-\lambda} = \sum_{\lambda=0}^{\kappa}\binom{\kappa}{\lambda}\sum_{i=1}^{D}t_i^{\kappa-\lambda}\sum_{j=1}^{d}\rho_j^\lambda = \\ & \textstyle\sum_{\lambda=0}^{\kappa}\binom{\kappa}{\lambda}\sum_{i=1}^{D}t_i^{\kappa-\lambda}p_\lambda(\mathbf{r}) = \sum_{\lambda=0}^{\kappa}\binom{\kappa}{\lambda}p_{\kappa-\lambda}(\mathbf{t})p_\lambda(\mathbf{r}) \end{aligned}$$

Substitute $(t_1+\rho_1,\dots,t_i+\rho_j,\dots,t_D+\rho_d)$ for $(y_1,\dots,y_{dD})$ in the Newton-Girard formulæ with $m=dD$ and combine them with identity (6). This yields an inductively defined family of functions $E_\ell(v_0,\mathbf{v}_\ell,u_0,\mathbf{u}_\ell)$:

Definition 1.

$$\begin{aligned} &E_0\big(v_0,(),u_0,()\big) := 1,\\ &E_\ell(v_0,\mathbf{v}_\ell,u_0,\mathbf{u}_\ell) := -(1/\ell)\textstyle\sum_{\kappa=1}^{\ell}E_{\ell-\kappa}(v_0,\mathbf{v}_{\ell-\kappa},u_0,\mathbf{u}_{\ell-\kappa})(\sum_{\lambda=0}^{\kappa}\binom{\kappa}{\lambda}v_{\kappa-\lambda}u_\lambda). \end{aligned}$$

For instance, $E_1(v_0,\mathbf{v}_1,u_0,\mathbf{u}_1) = -v_1u_0 - v_0u_1$. Now we can make the following statement.

Lemma 5. For all $\ell \geq 0$ the following identity holds:

$$\varepsilon_\ell(\mathbf{t},\mathbf{r}) = E_\ell\big(D,\mathbf{p}_\ell(\mathbf{t}),d,\mathbf{p}_\ell(\mathbf{r})\big). \tag{7}$$

Proof. We prove the lemma by induction on ℓ using the Newton-Girard formulæ on the induction step.

For $\ell=0$ one obtains $\varepsilon_0(\mathbf{t},\mathbf{r}) = 1 = E_0\big(D,\mathbf{p}_0(\mathbf{t}),d,\mathbf{p}_0(\mathbf{r})\big)$ immediately by the definitions.

For $\ell>0$, combining identity (4) with the Newton-Girard formulæ, where $(y_1,\dots,y_m)$ is replaced by $(t_1+\rho_1,\dots,t_i+\rho_j,\dots,t_D+\rho_d)$, one obtains

$$(-1)^\ell\varepsilon_\ell(\mathbf{t},\mathbf{r}) = (1/\ell)\sum_{\kappa=1}^{\ell}(-1)^{\kappa-1}(-1)^{\ell-\kappa}\varepsilon_{\ell-\kappa}(\mathbf{t},\mathbf{r})p_\kappa(t_1+\rho_1,\dots,t_i+\rho_j,\dots,t_D+\rho_d).$$

From this and identity (6) it follows that

$$\varepsilon_\ell(\mathbf{t},\mathbf{r}) = -(1/\ell)\sum_{\kappa=1}^{\ell}\varepsilon_{\ell-\kappa}(\mathbf{t},\mathbf{r})\sum_{\lambda=0}^{\kappa}\binom{\kappa}{\lambda}p_{\kappa-\lambda}(\mathbf{t})p_\lambda(\mathbf{r}). \tag{8}$$

Using the induction assumption $\varepsilon_{\ell-\kappa}(\mathbf{t},\mathbf{r}) = E_{\ell-\kappa}\big(D, \mathbf{p}_{\ell-\kappa}(\mathbf{t}), d, \mathbf{p}_{\ell-\kappa}(\mathbf{r})\big)$ one easily obtains $\varepsilon_\ell(\mathbf{t},\mathbf{r}) = E_\ell\big(D, \mathbf{p}_\ell(\mathbf{t}), d, \mathbf{p}_\ell(\mathbf{r})\big)$ by the definition. The lemma is proven. $\square$

Using the functions E_ℓ, one can symbolically compute $\varepsilon_\ell(\mathbf{t},\mathbf{r})$ for any $\ell > 0$. For instance, $\varepsilon_1(\mathbf{t},\mathbf{r}) = -d\,p_1(\mathbf{t}) - D\,p_1(\mathbf{r})$.

Now we are to combine Definition 1 and Lemma 4. It is expressed via the following definition and lemma.

Definition 2. $S_\ell(u_0, \mathbf{u}_\ell) := \sum_{\mathbf{t}\in T} E_\ell(D, \mathbf{p}_\ell(\mathbf{t}), u_0, \mathbf{u}_\ell)\alpha_{\mathbf{t}}$.

Lemma 6. If a polynomial P of degree d solves equation (3) and $d > \ell$ for some $\ell \geq 0$ then $S_\ell\big(d, \mathbf{p}_\ell(\mathbf{r})\big) = 0$.

Proof. By Lemma 5 and the definition of S_ℓ one has $\sum_{\mathbf{t}\in T} \varepsilon_\ell(\mathbf{t},\mathbf{r})\alpha_{\mathbf{t}} = S_\ell\big(d, \mathbf{p}_\ell(\mathbf{r})\big)$. By Lemma 4 one obtains the identity $S_\ell\big(d, \mathbf{p}_\ell(\mathbf{r})\big) = 0$. $\square$

Yet, from the point of view of bounding the degree d, Lemma 6 is too general. We will figure out the cases when for some non-negative integer number $L \geq 0$ the identities $S_\ell\big(d, \mathbf{p}_\ell(\mathbf{r})\big) = 0$, with $0 \leq \ell \leq L$, yield a non-zero univariate polynomial $Q(u_0)$ such that $Q(d) = 0$. To be more precise, we are looking for L such that $Q(u_0) = S_L(u_0, 0, \ldots, 0)$. For this we have a closer look at the functions $E_\ell(v_0, \mathbf{v}_\ell, u_0, \mathbf{u}_\ell)$. These functions are obviously polynomials in $v_0, \mathbf{v}_\ell, u_0, \mathbf{u}_\ell$. The total degree w.r.t. $v_0, \ldots, v_\ell$ and $u_0, \ldots, u_\ell$ is ℓ, however one can prove a more precise connection between the powers of the v- and u-variables:

Lemma 7. For any term with the monomial part $v_0^{i_0} \ldots v_\ell^{i_\ell} u_0^{j_0} \ldots u_\ell^{j_\ell}$ that occurs in $E_\ell(v_0, \mathbf{v}_\ell, u_0, \mathbf{u}_\ell)$ the following equation holds: $\sum_{\kappa=0}^{\ell} \kappa(i_\kappa + j_\kappa) = \ell$.

The proof follows by induction on ℓ.

Proof. For $\ell = 0$ the statement trivially holds: $0i_0 + 0j_0 = 0$.

For $\ell = 1$ we have $E_1 = -v_1u_0 - v_0u_1$. Thus, (i_0, i_1, j_0, j_1) ranges over $(0, 1, 1, 0)$ and $(1, 0, 0, 1)$, for the terms $-v_1u_0$ and $-v_0u_1$ respectively. Trivially, we have $0 + 1 \cdot 1 + 0 \cdot 1 + 0 \cdot 0 = 1$ and $0 \cdot 1 + 1 \cdot 0 + 0 \cdot 0 + 1 \cdot 1 = 1$.

Now, let the property hold for all $\ell' < \ell$. We use the recursive definition of $E_\ell(v_0, \mathbf{v}_\ell, u_0, \mathbf{u}_\ell)$. It is a sum of terms of $E_{\ell-\kappa}(v_0, \mathbf{v}_{\ell-\kappa}, u_0, \mathbf{u}_{\ell-\kappa})$ multiplied by $\binom{\kappa}{\lambda}v_{\kappa-\lambda}u_\lambda$. Therefore, it is enough to consider an arbitrary product of this form.

Let q be a term of $E_{\ell-\kappa}(v_0, \mathbf{v}_{\ell-\kappa}, u_0, \mathbf{u}_{\ell-\kappa})$.

Let q correspond to the degrees $(i_0, \ldots, i_{\ell-\kappa}, j_0, \ldots j_{\ell-\kappa})$. By the induction assumption $\sigma_{\ell-\kappa} := 0 \cdot i_0 + i_1 + 2i_2 + \ldots + (\ell-\kappa)i_{\ell-\kappa} + 0 \cdot j_0 + j_1 + 2j_2 + \ldots + (\ell-\kappa)j_{\ell-\kappa} = \ell-\kappa$. Therefore, for the sum of the powers in the product $qv_{\kappa-\lambda}u_\lambda$ one obtains $\sigma_\ell = \sigma_{\ell-\kappa} + (\kappa-\lambda)\cdot 1 + \lambda \cdot 1 = \ell - \kappa + (\kappa - \lambda) + \lambda = \ell$. $\square$

This property is used when one wants to give the complete list of all coefficients of the powers of the variables $\mathbf{u}_\ell$, when $E_\ell(v_0, \mathbf{v}_\ell, u_0, \mathbf{u}_\ell)$ is considered as a polynomial in $\mathbf{u}_\ell$.

Definition 3. $A_{\mathbf{i}_\ell}(v_0, \mathbf{v}_\ell, u_0)$ denotes the coefficient of $\mathbf{u}_\ell^{\mathbf{i}_\ell}$ in $E_\ell(v_0, \mathbf{v}_\ell, u_0, \mathbf{u}_\ell)$, that is $E_\ell(v_0, \mathbf{v}_\ell, u_0, \mathbf{u}_\ell) = \sum_{\mathbf{i}_\ell} A_{\mathbf{i}_\ell}(v_0, \mathbf{v}_\ell, u_0) \cdot \mathbf{u}_\ell^{\mathbf{i}_\ell}$.

Using Definition 3 it is easy to obtain the representation of $S_\ell(u_0, \mathbf{u}_\ell)$ as a polynomial in $\mathbf{u}_\ell$:

$$S_\ell(u_0, \mathbf{u}_\ell) = \sum_{\mathbf{i}_\ell} \Big(\sum_{\mathbf{t}\in T} \alpha_{\mathbf{t}} A_{\mathbf{i}_\ell}\big(D, \mathbf{p}_\ell(\mathbf{t}), u_0\big)\Big) \cdot \mathbf{u}_\ell^{\mathbf{i}_\ell}. \tag{9}$$

Indeed,

$$\begin{aligned} S_\ell(u_0, \mathbf{u}_\ell) = {} & \sum_{\mathbf{t}\in T} E_\ell(D, p_1(\mathbf{t}), \ldots, p_\ell(\mathbf{t}), u_0, \mathbf{u}_\ell)\alpha_{\mathbf{t}} = \\ & \sum_{\mathbf{t}\in T} \big(\sum_{\mathbf{i}_\ell} A_{\mathbf{i}_\ell}(D, p_1(\mathbf{t}), \ldots, p_\ell(\mathbf{t}), u_0)\mathbf{u}_\ell^{\mathbf{i}_\ell}\big)\alpha_{\mathbf{t}} = \\ & \sum_{\mathbf{i}_\ell} \mathbf{u}_\ell^{\mathbf{i}_\ell} \big(\sum_{\mathbf{t}\in T} A_{\mathbf{i}_\ell}(D, p_1(\mathbf{t}), \ldots, p_\ell(\mathbf{t}), u_0)\alpha_{\mathbf{t}}\big) \end{aligned}$$

In its turn, each of the $A_{\mathbf{i}_\ell}(v_0, \mathbf{v}_\ell, u_0)$ is a polynomial in u_0, where the corresponding coefficients of u_0^μ are denoted by $B_{\mathbf{i}_\ell,\mu}(v_0, \mathbf{v}_\ell)$. As one will see now, the coefficients $B_{\mathbf{0}_\ell,\mu}$ play a special role.

Lemma 8. Let $L > 0$ be such that for any $0 \le \ell \le L-1$ the polynomial $S_\ell(u_0, \mathbf{0}_\ell)$ is everywhere zero, and moreover, for $\mathbf{i}_\ell \ne \mathbf{0}_\ell$ and $\mu \le \ell$ there exist polynomials $H_{\mathbf{i}_L,\ell,\mu}(D, u_0)$, such that

$$A_{\mathbf{i}_L}\big(D, \mathbf{p}_L(\mathbf{t}), u_0\big) = \sum_{\ell=0}^{L-1} \sum_{\mu=0}^{\ell} H_{\mathbf{i}_L,\ell,\mu}(D, u_0) B_{\mathbf{0}_\ell,\mu}\big(D, \mathbf{p}_l(\mathbf{t})\big). \tag{10}$$

Then $S_L(u_0, \mathbf{u}_L) = S_L(u_0, \mathbf{0}_L)$ for all u_0.

Proof. Since for any $0 \le \ell \le L-1$ the polynomial $S_\ell(u_0, \mathbf{0}_\ell)$ is everywhere zero, it follows that for any $0 \le \mu \le \ell$ the coefficient of u_0^μ in $S_\ell(u_0, \mathbf{0}_\ell) = \sum_{\mathbf{t}\in T} \alpha_{\mathbf{t}} A_{\mathbf{0}_\ell}\big(D, \mathbf{p}_\ell(\mathbf{t}), u_0\big) = \sum_{\mathbf{t}\in T} \alpha_{\mathbf{t}} \sum_{\mu=0}^{l} B_{\mathbf{0}_\ell,\mu}\big(D, \mathbf{p}_\ell(\mathbf{t})\big)u_0^\mu$ vanishes:

$$\sum_{\mathbf{t}\in T} \alpha_{\mathbf{t}} B_{\mathbf{0}_\ell,\mu}\big(D, \mathbf{p}_\ell(\mathbf{t})\big) = 0 \text{ for all } 0 \le \mu \le \ell. \tag{11}$$

Now, plugging identity (10) into identity (9), we obtain that for any $\mathbf{i}_\ell \ne \mathbf{0}_\ell$ the coefficient of $\mathbf{u}_\ell^{\mathbf{i}_\ell}$ in $S_L(u_0, \mathbf{u}_L)$ vanishes. Indeed, it is equal to

$$\begin{aligned} & \sum_{\mathbf{t}\in T} \alpha_{\mathbf{t}} \sum_{\ell=0}^{L-1} \sum_{\mu=0}^{\ell} H_{\mathbf{i}_L,\ell,\mu}(D, u_0) B_{\mathbf{0}_\ell,\mu}\big(D, \mathbf{p}_\ell(\mathbf{t})\big) = \\ & \sum_{\ell=0}^{L-1} \sum_{\mu=0}^{\ell} H_{\mathbf{i}_L,\ell,\mu}(D, u_0) \sum_{\mathbf{t}\in T} \alpha_{\mathbf{t}} B_{\mathbf{0}_\ell,\mu}\big(D, \mathbf{p}_\ell(\mathbf{t})\big) =^{\text{identity (11)}} \\ & \sum_{\ell=0}^{L-1} \sum_{\mu=0}^{\ell} H_{\mathbf{i}_L,\ell,\mu}(D, u_0) \cdot 0 = 0. \end{aligned}$$

Therefore, $S_L(u_0, \mathbf{u}_\ell) = S_L(u_0, \mathbf{0}_\ell)$. $\square$

Lemma 9 (Framework)**.** Let the set $\mathcal{L} := \{\ell | S_\ell(u_0, \mathbf{0}_\ell) \text{ is not everywhere zero}\}$ be non-empty and $L = \min(\mathcal{L})$. Moreover, let for all $\mathbf{i}_\ell \ne \mathbf{0}_\ell$ and $\mu \le \ell < L$ there exist polynomials $H_{\mathbf{i}_L,\ell,\mu}(D, u_0)$ such that identities (10) hold. Then either $d \le \max\{L, \deg(G_0)/(D-1)\}$, or d is a root of $Q(u_0) := S_L(u_0, \mathbf{0}_L)$.

Proof. If $d > L$ and $d > \deg(G_0)/(D-1)$, then $dD - L > d(D-1) > \deg(G_0)$ which implies that the coefficient of x^{dD-L} on the l.h.s. of equation (3) must vanish. We apply Lemma 6 to obtain $S_L\big(d, \mathbf{p}_L(\mathbf{r})\big) = 0$. Next, we apply Lemma 8 and obtain $S_L(u_0, \mathbf{u}_\ell) = S_L(u_0, \mathbf{0}_\ell)$ for all u_0. From this and the condition $S_L\big(d, \mathbf{p}_L(\mathbf{r})\big) = 0$, it follows that $Q(d) = 0$. □

3. Existence of a degree polynomial for $0 \leq \ell \leq 5$

It turned out that a property stronger than identity (10) holds for $E(v_0, \mathbf{v}_\ell, u_0, \mathbf{u}_\ell)$, where $1 \leq \ell \leq 5$. It is stated in the following lemma.

Lemma 10. For all $1 \leq L \leq 5$, for all $\mathbf{i}_L \neq \mathbf{0}_L$ and $\mu \leq \ell < L$ there exist polynomials $H_{\mathbf{i}_L,\ell,\mu}(v_0, u_0)$ such that $A_{\mathbf{i}_L}(v_0, \mathbf{v}_L, u_0) = \sum_{\ell=0}^{L-1} \sum_{\mu=0}^{\ell} H_{\mathbf{i}_L,\ell,\mu}(v_0, u_0) B_{\mathbf{0}_\ell,\mu}(v_0, \mathbf{v}_\ell)$. [2]

Proof. The coefficients $A_{\mathbf{i}_L}(v_0, \mathbf{v}_L, u_0)$, $B_{\mathbf{0}_\ell,\mu}(v_0, \mathbf{v}_\ell)$ and $H_{\mathbf{i}_L,\ell,\mu}(v_0, u_0)$ are computed symbolically for all $0 \leq L \leq 5$, $\mu \leq \ell < L$ in the script `lemma-7.mw` (see Section 1 for the url). Linear algebra suffices to obtain $H_{\mathbf{i}_L,\ell,\mu}(v_0, u_0)$. [3] Fix some $1 \leq L \leq 5$. Think of $A_{\mathbf{i}_L}(v_0, \mathbf{v}_L, u_0)$, and $B_{\mathbf{0}_\ell,\mu}(v_0, \mathbf{v}_\ell)$ as polynomials in $v_1, ..., v_L$ only, but with coefficients which belong to the field $\mathbb{K}(u_0, v_0)$ of rational functions in u_0, v_0 over the field of constants $\mathbb{K}$. Let $H_{\mathbf{i}_L,\ell,\mu}(v_0, u_0)$ belong to $\mathbb{K}(u_0, v_0)$. Make the list of all the monomials in $v_1, ..., v_L$ of degree $\leq L$. Represent each of $A_{\mathbf{i}_L}(v_0, \mathbf{v}_L, u_0)$ and $B_{\mathbf{0}_\ell,\mu}(v_0, \mathbf{v}_\ell)$-s by the corresponding vectors $F = (F_{j_\ell})$ and $(F_{j_\ell,\ell,\mu})$ of their coefficients w.r.t. these monomials, with $\mu \leq \ell < L$. Direct computation shows that the linear system $M\mathbf{H} = F$ is solvable over $\mathbb{K}(u_0, v_0)$, where M is the matrix with columns $F_{\ell,\mu}$, and j_ℓ ranges over rows. Moreover, the entries of the solution $\mathbf{H} = \big(H_{\mathbf{i}_L,\ell,\mu}(v_0, u_0)\big)$ are polynomials.

The polynomials $B_{\mathbf{0}_\ell,\mu}(v_0, \mathbf{v}_\ell)$ and the corresponding coefficients $H_{\mathbf{i}_L,\ell,\mu}(v_0, u_0)$ are given in the Appendix, Section 8.1. □

To provide the reader with intuition behind our constructions, we consider the cases $d > 0, 1, 2$ in more detail.

3.1. $d > 0$

In this case $Dd > d$. Therefore, we have to cancel x^{Dd} on the left-hand side of equation 3, that is $S_0(d) = 0$ by Lemma 6. Using the definition $S_0(u_0) = \sum_{\mathbf{t}\in T} E_0(D, u_0)$ and the definition $E_0(v_0, u_0) = 1$ we obtain $S_0(u_0) = \sum_{\mathbf{t}\in T} \alpha_\mathbf{t}$, and therefore

$$S_0(d) = \sum_{\mathbf{t}\in T} \alpha_\mathbf{t} = 0. \tag{12}$$

If the sum $\sum_{\mathbf{t}\in T} \alpha_\mathbf{t} \neq 0$ then the coefficient at x^{Dd} on the l.h.s. does not vanish and the assumption $d > 0$ cannot hold. (The bound on the degree is found: $d = 0$.) If this sum vanishes then continue to check for $d > 1$.

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[2] This identity is stronger than identity (10) because it holds *for all* v_0 and $\mathbf{v}_\ell$, not only for $v_0 := D$ and $\mathbf{v}_\ell := \mathbf{p}_\ell(\mathbf{t})$.

[3] This observation belongs to M. Petkovšek. Originally we used the procedure Groebner[NormalForm] to perform the division of the polynomial $A_{\mathbf{i}_L}(v_0, \mathbf{v}_L, u_0)$ in $\mathbf{v}_L$ by the polynomials from the set $\{B_{(),0} = 1\} \cup \{B_{\mathbf{0}_\ell,\mu}(v_0, \mathbf{v}_l)\}_{1\leq \ell < L,\ 1 \leq \mu \leq l}$. Note that $B_{\mathbf{0}_\ell,0}(v_0, \mathbf{v}_\ell) = 0$ for $\ell > 0$.

3.2. $d > 1$

Again comparing the l.h.s. and the r.h.s. of equation 3 we have $Dd - 1 > Dd - d = (D-1)d \geq d$, so we have to cancel x^{Dd-1} as well. By Lemma 6 this means that $S_1\big(d, p_1(\mathbf{r})\big) = 0$. We simplify this equation, using condition (12). By the definition, $S_1(u_0, u_1) := \sum_{\mathbf{t}\in T} E_1(D, p_1(\mathbf{t}), u_0, u_1)\alpha_{\mathbf{t}}$. By the definition, $E_1(v_0, v_1, u_0, u_1) = -v_0u_1 - v_1u_0$. From what follows that

$$\begin{aligned} S_1(u_0, u_1) &= -Du_1 \textstyle\sum_{\mathbf{t}\in T} \alpha_{\mathbf{t}} - u_0 \sum_{\mathbf{t}\in T} p_1(\mathbf{t})\alpha_{\mathbf{t}} \\ &=^{\text{equation 12}} -u_0 \textstyle\sum_{\mathbf{t}\in T} p_1(\mathbf{t})\alpha_{\mathbf{t}} \\ &= S_1(u_0, 0) \end{aligned} \tag{13}$$

If the sum $\sum_{\mathbf{t}\in T} p_1(\mathbf{t})\alpha_{\mathbf{t}}$ does not vanish then $S_1(u_0, 0) = 0$ only if $u_0 = 0$, which contradicts $d > 1$. (Then the bound is found $d \leq 1$.) If the sum $\sum_{\mathbf{t}\in T} p_1(\mathbf{t})\alpha_{\mathbf{t}}$ vanishes, then $S_1(u_0, 0)$ is everywhere zero and we continue to check for $d > 2$.

3.3. $d > 2$

Comparing the l.h.s. and the r.h.s. of equation 3 we have $Dd-2 > Dd-d = (D-1)d \geq d$, so we have to cancel n^{Dd-2}. By Lemma 6 this means that $S_2\big(d, p_1(\mathbf{r}), p_2(\mathbf{r})\big) = 0$. We are to simplify this equation, using conditions (12) and (13). By the definition, $S_2(u_0, u_1, u_2) = \sum_{\mathbf{t}\in T} E_2(D, p_1(\mathbf{t}), p_2(\mathbf{t}), u_0, u_1, u_2)\alpha_{\mathbf{t}}$. One unfolds the recursive definition of $E_2(v_0, v_1, v_2, u_0, u_1, u_2)$ and obtains the following coefficients $A_{(i_1,i_2)}(v_0, v_1, v_2, u_0)$ at $u_1^{i_1}u_2^{i_2}$, see Appendix and Maple script as well:

- $A_{(0,0)}(v_0, v_1, v_2, u_0) = (1/2)u_0^2v_1^2 - (1/2)u_0v_2$ is the u_1, u_2-free term of E_2,
- $A_{(1,0)}(v_0, v_1, v_2, u_0) = u_0v_0v_1 - v_1$,
- $A_{(2,0)}(v_0, v_1, v_2, u_0) = (1/2)v_0^2$,
- $A_{(0,1)}(v_0, v_1, v_2, u_0) = -(1/2)v_0$.

Consequently, the corresponding coefficients of $S_2(u_0, u_1, u_2)$ are

- $(u_0/2)\sum_{\mathbf{t}\in T}(u_0p_1(\mathbf{t})^2 - p_2(\mathbf{t})\alpha_{\mathbf{t}}$ of $u_1^0u_2^0$,
- $(u_0D - 1)\sum_{\mathbf{t}\in T} p_1(\mathbf{t})\alpha_{\mathbf{t}} = (1/u_0 - D)S_1(u_0, 0)$ of $u_1^1u_2^0$,
- $(1/2)D^2\sum_{\mathbf{t}\in T}\alpha_{\mathbf{t}} = (1/2)D^2S_0(u_0, 0)$ of $u_1^2u_2^0$,
- $-(1/2)DS_0(u_0, 0)$ of $u_1^0u_2^1$.

Since $S_1(u_0, 0)$ and $S_0(u_0)$ are everywhere zero, we immediately obtain that the coefficients of $u_1^{i_1}u_2^{i_2}$ vanish for $(i_1, i_2) \neq (0, 0)$, and therefore $S_2(u_0, u_1, u_2) = S_2(u_0, 0, 0)$. Thus,

$$S_2(d, 0, 0) = (d/2)\sum_{\mathbf{t}\in T}(dp_1(\mathbf{t})^2 - p_2(\mathbf{t}))\alpha_{\mathbf{t}} = 0 \tag{14}$$

If $S_2(u_0, 0, 0) = \sum_{\mathbf{t}\in T}\big(u_0p_1^2(\mathbf{t}) - p_2(\mathbf{t})\big)\alpha_{\mathbf{k}}$ is not a constant zero, then d is the root of the polynomial $S_2(u_0, 0, 0)$. The root $d = 0$ does not make sense for $d > 2$, therefore one has to consider only the root

$$d = \frac{\sum_{\mathbf{t}\in T} p_2(\mathbf{t})\alpha_{\mathbf{t}}}{\sum_{\mathbf{t}\in T} p_1^2(\mathbf{t})\alpha_{\mathbf{t}}}$$

If the computed value of d is not a non-negative integer, it means that $d > 2$ cannot hold (and the bound is found $d \leq 2$.

If $S_2(u_0, 0, 0)$ is everywhere zero, we continue to check for $d > 3$.

3.4. $d > 3$

Comparing the l.h.s. and the r.h.s. of equation (3) we have $Dd-3 > Dd-d = (D-1)d \geq d$, so we have to cancel n^{Dd-3}. By Lemma 6 this means that $S_3(d, p_1(\mathbf{r}), p_2(\mathbf{r}), p_3(\mathbf{r})) = 0$. We are to simplify this equation, using conditions (12), (13) and (14). By the definition,

$$S_3(u_0, u_1, u_2, u_3) = \sum_{\mathbf{t}\in T} E_3\big((D, p_1(\mathbf{t}), p_2(\mathbf{t}), p_3(\mathbf{t}), u_0, (u_1, u_2, u_3)\big)\alpha_{\mathbf{t}}$$

One unfolds the recursive definition of $E_3(\mathbf{v}_3, u_0, \mathbf{u}_3)$ and obtains the coefficients $A_{\mathbf{i}_3}(\mathbf{v}_3, u_0)$ at $u_1^{i_1}u_2^{i_2}u_3^{i_3}$. The polynomial $A_{\mathbf{0}_3}(\mathbf{v}_3, u_0)$ and the coefficients $H_{\mathbf{i}_3\ell\mu}(v_0, u_0)$ for the remaining $A_{\mathbf{i}_3}(\mathbf{v}_3, u_0)$ are given in Appendix (Section 8.1).

Further, using the conditions 12, 13 and 14 we obtain, that

$$S_3(u_0, \mathbf{u}_3) = \textstyle\sum_{\mathbf{t}\in T}\Big(-(1/6)p_1^3(\mathbf{t})u_0^3 + (1/2)p_1(\mathbf{t})p_2(\mathbf{t})u_0^2 - (1/3)p_3(\mathbf{t})u_0\Big)\alpha_{\mathbf{t}}.$$

If the polynomials $S_0(u_0)$, $S_1(u_0, 0)$ and $S_2(u_0, 0, 0)$ are everywhere zero, then $S_3(d, 0, 0, 0) = 0$.If $S_3(u_0, 0, 0, 0)$ is not everywhere zero then d must be amongst its (natural) roots. If $S_3(u_0, 0, 0, 0)$ is everywhere zero as well, continue to check for $d > 4$ etc.

The main theorem below gives an effective bound on d in the case when there exists $0 \leq L \leq 5$ such that $S_L(u_0, \mathbf{0}_L)$ is not everywhere zero.

Theorem 4. *If the set $\mathcal{L} = \{\ell | S_\ell(u_0, \mathbf{0}_\ell)$ is not everywhere zero$\}$ is not empty and, moreover, $L := \min(\mathcal{L}) \leq 5$, then either $d \leq \max\{L, \deg(G_0)/(D-1)\}$, or d must be among the non-negative integer roots of $S_L(u_0, \mathbf{0}_L)$.*

Proof. The condition $L \leq 5$ together with Lemma 10 yields the conditions of the framework lemma. Applying it straightforwardly gives the conclusion of the theorem. □

Corollary 1. For any difference equation (3) with $D = 2$ and $\tau_i = a + i - 1$ where $i = 1, 2, 3$, there is $0 \leq L \leq 5$ such that $S_L(u_0, \mathbf{0}_L)$ is not everywhere zero. Therefore, the degree d of a polynomial solution P either does not exceed $\max\{L, \deg(G_0)/(D-1)\}$, or must be among the non-negative integer roots of the polynomial $S_L(u_0, \mathbf{0}_L)$.

Proof. Without lost of generality one can consider only the case $a = 0$. Indeed, if $G\big(P(x-a), P(x-a-1), P(x-a-2)\big) + G_0(x) = 0$ has a polynomial solution $P(x)$ then $G\big(F(x), F(x-1), F(x-2)\big) + G_0(x) = 0$ has the polynomial solution $F(x) = P(x-a)$, of the same degree.

Now, for $a = 0$ assume the opposite: for all $0 \leq \ell \leq 5$ the polynomials $S_\ell(u_0, \mathbf{0}_\ell)$ for $G\big(P(x), P(x-1), P(x-2)\big) + G_0(x) = 0$ are everywhere zero. We show that in this case G_D is reduced to the zero polynomial. With $D = 2$ and $\tau_i = i$, where $i = 0, 1, 2$, one has $T = \{(i_1, i_2) | 0 \leq i_1 \leq i_2 \leq 2\}$. Compute the concrete values of $B_{\mathbf{0}_\ell,\mu}\big(D, \mathbf{p}_\ell(\mathbf{t})\big)$ for all $\mathbf{t} \in T$, $1 \leq \ell \leq 5$, $1 \leq \mu \leq \ell$, and $B_{(),0}$. These values form the matrix of the overdetermined linear system of 16 equations $\sum_{\mathbf{t}\in T} B_{\mathbf{0}_\ell,\mu}\big(D, \mathbf{p}_\ell(\mathbf{t})\big)\alpha_{\mathbf{t}} = 0$ for 6 variables $\alpha_{\mathbf{t}}$ (see Appendix, Section 8.2 for more detail). This system has only the zero solution $\bar{\alpha} = \mathbf{0}_6$ which means that the polynomial G_D is everywhere zero which contradicts the condition $D = 2$. □

In the same way one proves the similar statement for difference equations with $D = 3$ and $\tau_i = a + i - 1$, where $i = 1, 2$.

Corollary 2. For any difference equation (3) with $D = 3$ and $\tau_i = a+i-1$ with $i = 1, 2$, there is $0 \le L \le 5$ such that $S_L(u_0, \mathbf{0}_L)$ is not everywhere zero. Therefore, the degree d of a polynomial solution P either does not exceed the number $\max\{L, \deg(G_0)/(D-1)\}$, or must be among the non-negative integer roots of the polynomial $S_L(u_0, \mathbf{0}_L)$.

Proof. As in the corollary 1 we consider $a = 0$. Assume the opposite: for all $0 \le \ell \le 5$ the polynomials $S_\ell(u_0, \mathbf{0}_\ell)$ are everywhere zero. We will show that in this case G_D is reduced to a zero polynomial. Take

$$T = \{(\tau_1, \tau_1, \tau_1), (\tau_1, \tau_1, \tau_2), (\tau_1, \tau_2, \tau_2), (\tau_2, \tau_2, \tau_2)\},$$

and compute $B_{\mathbf{0}_\ell,\mu}(D, p_1(\mathbf{t}), \ldots, p_\ell(\mathbf{t}))$ for all $\mathbf{t} \in T$, $1 \le \ell \le 5$, $1 \le \mu \le l$, and $B_{(),0}$. Out of the conditions

$$\sum_{\mathbf{t}\in T} B_{\mathbf{0}_\ell,\mu}\big(D, p_1(\mathbf{t}), \ldots, p_\ell(\mathbf{t})\big)\alpha_{\mathbf{t}} = 0$$

obtain the over-defined system of 16 linear equations w.r.t. 4 variables $\alpha_{\mathbf{t}}$. The matrix looks like follows:

$$\begin{pmatrix}
1 & 1 & 1 & 1 \\
0 & -1 & -2 & -3 \\
0 & -1/2 & -1 & -3/2 \\
0 & 1/2 & 2 & 9/2 \\
0 & -1/3 & -2/3 & -1 \\
0 & 1/2 & 2 & 9/2 \\
0 & -1/6 & -4/3 & -9/2 \\
0 & -1/4 & -1/2 & -3/4 \\
0 & 11/24 & 11/6 & 33/8 \\
0 & -1/4 & -2 & -27/4 \\
0 & 1/24 & 2/3 & 27/8 \\
0 & -1/5 & -2/5 & -3/5 \\
0 & 5/12 & 5/3 & 15/4 \\
0 & -7/24 & -7/3 & -63/8 \\
0 & 1/12 & 4/3 & 27/4 \\
0 & -1/120 & -4/15 & -81/40
\end{pmatrix}$$

The matrix is computed and the system is solved in `corollaries_Ds.mw`, archived in the tar-ball. The system has only the trivial solution, $\alpha_{\mathbf{t}} = 0$ for all $\mathbf{t} \in T$, $1 \le \ell \le 5$, $1 \le \mu \le l$ and $B_{(),0} = 1$, so the recurrence relation degenerates to a linear recurrence relation with $D' = D - 1$. $\square$

Now consider what happens if the conditions of Theorem 4 do not hold, that is, for all $0 \le \ell \le 5$ the polynomials $S_\ell(u_0, \mathbf{0}_\ell)$ are everywhere zero. The next lemma shows that then, in general, $S_6(u_0, \mathbf{u}_6)$ and $S_6(u_0, \mathbf{0}_6)$ do not have to be equal as polynomials and therefore $S_6(u_0, \mathbf{0}_6)$ cannot be taken as a degree polynomial.

Lemma 11. If the polynomial $S_\ell(u_0, \mathbf{0}_\ell)$ is everywhere zero for any $0 \le \ell \le 5$, then

$$S_6(u_0, \mathbf{u}_6) = S_6(u_0, \mathbf{0}_6) + (1/8)(u_1^2 - u_2 u_0) \sum_{\mathbf{t} \in T} p_2^2(\mathbf{t}) \alpha_{\mathbf{t}}. \tag{15}$$

Proof. The computations of $H_{\mathbf{i}_6,\ell,\mu}$ are performed as in the proof of Lemma 10. The coefficients $H_{\mathbf{i}_6,\ell,\mu}$ for $A_{\mathbf{i}_6}(v_0, \mathbf{v}_6, u_0)$ can be found using linear algebra, except those for $A_{(0,1,0,0,0,0)}$ and $A_{(2,0,0,0,0,0)}$. Since, as in the proof of Lemma 8, all $\sum_{\mathbf{t}\in T} B_{\mathbf{0}_\ell,\mu}\big(D, \mathbf{p}_\ell(\mathbf{t})\big)\alpha_{\mathbf{t}}$ vanish, the sums $\sum_{\mathbf{t}\in T} A_{\mathbf{i}_6}\big(D, \mathbf{p}_6(\mathbf{t}), u_0\big)\alpha_{\mathbf{t}}$ vanish as well, if $\mathbf{i}_6 \neq (0,1,0,0,0,0)$ and $\mathbf{i}_6 \neq (2,0,0,0,0,0)$.

The linear systems of the form $M\mathbf{H} = F$ for $A_{(0,1,0,0,0,0)}$ and $A_{(2,0,0,0,0,0)}$ are not solvable over $\mathbb{K}(u_0, v_0)$. However, replacing $B_{\mathbf{0}_2,1}(v_1, v_2)$ with v_2^2 in the list of polynomials $\{B_{(),0} = 1\} \cup \{B_{\mathbf{0}_\ell,\mu}(v_0, \mathbf{v}_2)\}_{1\le \ell \le 5, 1 \le \mu \le \ell}$ one can obtain the alternative systems $M'\mathbf{H} = F$ which are solvable for $A_{(0,1,0,0,0,0)}(v_0, \mathbf{v}_6, u_0)$ and $A_{(2,0,0,0,0,0)}(v_0, \mathbf{v}_6, u_0)$ over $\mathbb{K}(u_0, v_0)$. The coefficients of v_2^2 are $-(1/8)u_0$ and $1/8$ respectively. Identity (15) follows from these identities and the definition of $S_\ell(u_0, \mathbf{u}_\ell)$. Check `lemma-8.mw` from `nonlindifeq.tar.gz` archive mentioned in the Introduction, where all $H_{\mathbf{i}_6,\ell,\mu}$, for the original $M\mathbf{H} = F$ and the alternative $M'\mathbf{H} = F$ systems are computed. □

Therefore, if the polynomials $S_\ell(u_0, \mathbf{0}_\ell)$ are everywhere zero for all $0 \le \ell \le 5$ then the proposed approach, in general, does not give a bound on d.

Now, the natural question is if it is possible at all, that there exists a difference equation for which $S_\ell(u_0, \mathbf{0}_\ell) \equiv 0$ for all $0 \le \ell \le 5$ and therefore we cannot give a bound on the degree d of a possible polynomial solution. The answer is "yes" and an example of such a difference equation is $P(x-1)P(x-2)P(x-4) - 2P(x-1)P(x-3)P(x-3) + P(x-1)P(x-3)P(x-4) + P(x-2)P(x-2)P(x-3) - 2P(x-2)P(x-2)P(x-4) + P(x-2)P(x-3)P(x-3) = 0$. It is a routine to check that $S_\ell(u_0, \mathbf{0}_\ell) \equiv 0$ for all $\ell = 0 \ldots 5$. Moreover $\sum_{\mathbf{t}\in T} p_2^2(\mathbf{t})\alpha_{\mathbf{t}} = 16$, which means by lemma 11 that $S_6\big(d, p_1(\mathbf{r}), p_2(\mathbf{r})\big) = 0$ is reduced to $S_6(d, \mathbf{0}_6) - 2dp_2(\mathbf{r}) + 2p_1^2(\mathbf{r}) = 0$ that is dependency on the solution's roots does not vanish. This example can be generalised by the following statement

Corollary 3. For any difference equation with $D = 3$ and $\tau_i = a + i - 1$ with $i = 1..4$, the polynomials $S_\ell(u_0, \mathbf{0}_\ell)$ are everywhere zeros for all $0 \le \ell \le 5$, if and only if the coefficients of G_D satisfy

$$\begin{aligned}
\alpha_{111} &= x_8 + x_2 + 6x_3 + x_4 + 6x_5 + 21x_6 + 56x_7 \\
\alpha_{112} &= -6x_8 - 6x_2 - 35x_3 - 6x_4 - 35x_5 - 120x_6 - 315x_7 \\
\alpha_{113} &= 6x_8 + 4x_2 + 24x_3 + 3x_4 + 20x_5 + 70x_6 + 180x_7 \\
\alpha_{114} &= -2x_8 - x_2 - 6x_3 - 3x_5 - 12x_6 - 30x_7 \\
\alpha_{122} &= 9x_8 + 11x_2 + 60x_3 + 12x_4 + 64x_5 + 210x_6 + 540x_7 \\
\alpha_{123} &= -18x_8 - 13x_2 - 72x_3 - 12x_4 - 66x_5 - 216x_6 - 540x_7 \\
\alpha_{124} &= 6x_8 + 6x_2 + 25x_3 + 23x_5 + 66x_6 + 153x_7 + x_1 + 6x_4 \\
\alpha_{133} &= 9x_8 - 2x_1 - 3x_2 - 9x_4 - 16x_5 - 27x_6 - 54x_7 \\
\alpha_{134} &= -6x_8 + 2x_2 + 3x_3 + 13x_5 + 28x_6 + 63x_7 + x_1 + 6x_4 \\
\alpha_{144} &= x_8 \\
\alpha_{222} &= -6x_2 - 27x_3 - 36x_5 - 108x_6 - 270x_7 - 8x_4 \\
\alpha_{223} &= 12x_2 + 45x_3 + 63x_5 + 171x_6 + 405x_7 + x_1 + 18x_4 \\
\alpha_{224} &= -8x_2 - 24x_3 - 34x_5 - 84x_6 - 189x_7 - 2x_1 - 12x_4 \\
\alpha_{233} &= x_1 \\
\alpha_{234} &= x_2 \\
\alpha_{244} &= x_3 \\
\alpha_{333} &= x_4 \\
\alpha_{334} &= x_5 \\
\alpha_{344} &= x_6 \\
\alpha_{444} &= x_7
\end{aligned} \tag{16}$$

for some real numbers $x_1, \ldots, x_8$. In this case the sum $\sum_{\mathbf{t}\in T} p_2^2(\mathbf{t})\alpha_{\mathbf{t}}$ is equal to $g(x_1, \ldots, x_7) = 48x_2 + 144x_3 + 96x_4 + 240x_5 + 576x_6 + 1296x_7 + 16x_1$ and if $x_1, \ldots, x_7$ are such that $g(x_1, \ldots, x_7) \neq 0$, then $S_6(u_0, \mathbf{u}_6) \neq S_6(u_0, \mathbf{0}_6)$.

Proof. The proof is technically similar to the proof of Corollary 1. We construct a linear system w.r.t. $\bar{\alpha}$, which has solutions if and only if all $S_\ell(u_0, \mathbf{0}_\ell) \equiv 0$ with $0 \leq \ell \leq 6$. For $D = 3$ and $\tau_i = a + i - 1$ with $i = 1..4$ we have that $T =$

$$\{(1,1,1),(1,1,2),(1,1,3),(1,1,4),(1,2,2),(1,2,3),(1,2,4),(1,3,3),(1,3,4),(1,4,4),$$
$$(2,2,2),(2,2,3),(2,2,4),(2,3,3),(2,3,4),(2,4,4),(3,3,3),(3,3,4),(3,4,4),(4,4,4)\}$$

Out of the conditions

$$\sum_{\mathbf{t}\in T} B_{\ell\mu}\big(D, p_1(\mathbf{t}), \ldots, p_\ell(\mathbf{t})\big)\alpha_{\mathbf{t}} = 0$$

obtain the over-defined homogeneous system of 16 linear equations w.r.t. 10 variables $\alpha_{\mathbf{t}}$, with the matrix, given below in two parts (the first part gives columns 1-10, the second part gives columns 11-10):

1	1	1	1	1	1	1	1	1	1
−3	−4	−5	−6	−5	−6	−7	−7	−8	−9
−3/2	−3	−11/2	−9	−9/2	−7	−21/2	−19/2	−13	−33/2
9/2	8	25/2	18	25/2	18	49/2	49/2	32	81/2
−1	−10/3	−29/3	−22	−17/3	−12	−73/3	−55/3	−92/3	−43
9/2	12	55/2	54	45/2	42	147/2	133/2	104	297/2
−9/2	−32/3	−125/6	−36	−125/6	−36	−343/6	−343/6	−256/3	−243/2
−3/4	−9/2	−83/4	−129/2	−33/4	−49/2	−273/4	−163/4	−169/2	−513/4
33/8	107/6	1523/24	345/2	923/24	193/2	5411/24	4163/24	1979/6	4185/8
−27/4	−24	−275/4	−162	−225/4	−126	−1029/4	−931/4	−416	−2673/4
27/8	32/3	625/24	54	625/24	54	2401/24	2401/24	512/3	2187/8
−3/5	−34/5	−49	−1026/5	−13	−276/5	−1057/5	−487/5	−1268/5	−2049/5
15/4	28	1883/12	585	267/4	231	2933/4	5513/12	3224/3	7455/4
−63/8	−134/3	−4715/24	−639	−2915/24	−363	−23569/24	−18361/24	−4972/3	−23733/8
27/4	32	1375/12	324	375/4	252	2401/4	6517/12	3328/3	8019/4
−81/40	−128/15	−625/24	−324/5	−625/24	−324/5	−16807/120	−16807/120	−4096/15	−19683/40

1	1	1	1	1	1	1	1	1	1
−6	−7	−8	−8	−9	−10	−9	−10	−11	−12
−6	−17/2	−12	−11	−29/2	−18	−27/2	−17	−41/2	−24
18	49/2	32	32	81/2	50	81/2	50	121/2	72
−8	−43/3	−80/3	−62/3	−33	−136/3	−27	−118/3	−155/3	−64
36	119/2	96	88	261/2	180	243/2	170	451/2	288
−36	−343/6	−256/3	−256/3	−243/2	−500/3	−243/2	−500/3	−1331/6	−288
−12	−113/4	−72	−89/2	−353/4	−132	−243/4	−209/2	−593/4	−192
66	3275/24	856/3	1355/6	3217/8	1846/3	2673/8	3227/6	18683/24	1056
−108	−833/4	−384	−352	−2349/4	−900	−2187/4	−850	−4961/4	−1728
54	2401/24	512/3	512/3	2187/8	1250/3	2187/8	1250/3	14641/24	864
−96/5	−307/5	−1088/5	−518/5	−1299/5	−416	−729/5	−302	−2291/5	−3072/5
120	3835/12	896	1750/3	5091/4	2136	3645/4	5141/3	32279/12	3840
−252	−14497/24	−4288/3	−3436/3	−18261/8	−11660/3	−15309/8	−10235/3	−130493/24	−8064
216	5831/12	1024	2816/3	7047/4	3000	6561/4	8500/3	54571/12	6912
−324/5	−16807/120	−4096/15	−4096/15	−19683/40	−2500/3	−19683/40	−2500/3	−161051/120	−10368/5

This system has solutions of the form (16).

$\square$

However, $D = 2$ gives a special case where the framework lemma is applicable for $\ell = 6$.

Corollary 4. For all difference equations with $D = 2$, if $S_\ell(u_0, \mathbf{0}_\ell)$ is everywhere zero for $0 \le \ell \le 5$ then $S_6(u_0, \mathbf{u}_6) = S_6(u_0, \mathbf{0}_6)$. From this it follows that if $S_6(u_0, \mathbf{0}_6)$ is not everywhere zero, then either $d \le \max\{6, \deg(G_0)\}$, or d is one of the positive integer roots of $S_6(u_0, \mathbf{0}_6)$ if they exist.

Proof. Recall lemma 11:

$$S_6(u_0, \mathbf{u}_6) = S_6(u_0, \mathbf{0}_6) - u_2 \cdot (1/8)u_0 \sum_{\mathbf{t}\in T} p_2^2(\mathbf{t})\alpha_{\mathbf{t}} + u_1^2 \cdot (1/8) \sum_{\mathbf{t}\in T} p_2^2(\mathbf{t})\alpha_{\mathbf{t}}$$

It is easy to see that to prove the corollary one just need to prove $\sum_{\mathbf{t}\in T} p_2^2(\mathbf{t})\alpha_{\mathbf{t}} = 0$. We show that for $D = 2$ this follows from $S_4(u_0, \mathbf{0}_4) \equiv 0$. For this we consider the coefficients $B_{\mathbf{0}_4,3}$, $B_{\mathbf{0}_4,2}$ and $B_{\mathbf{0}_4,1}$ with $\mathbf{t} = (t_1, t_2)$:

- $B_{\mathbf{0}_4,3} = -(1/4)p_2(\mathbf{t})p_1^2(\mathbf{t}) =(-1/4)p_2(\mathbf{t}) \cdot (t_1 + t_2)^2 = (-1/4)p_2(\mathbf{t}) \cdot (t_1^2 + t_2^2 + 2t_1t_2) = (-1/4)p_2(\mathbf{t})(t_1^2+t_2^2)-(1/4)p_2(\mathbf{t})\cdot 2t_1t_2 =(-1/4)p_2^2(\mathbf{t})-(1/2)(t_1^3t_2+t_1t_2^3) =(-1/4)p_2^2(\mathbf{t})-(1/2)y_{\mathbf{t}}$, where $y_{\mathbf{t}}$ denotes $t_1^3t_2 + t_1t_2^3$;
- in $B_{\mathbf{0}_4,2} = (1/3)p_3(\mathbf{t}) \cdot p_1(\mathbf{t}) + (1/8)p_2^2(\mathbf{t})$ we first pay attention to $p_3(\mathbf{t})p_1(\mathbf{t}) =(t_1^3 + t_2^3)(t_1 + t_2) =t_1^4 + t_2^4 + t_1^3t_2 + t_1t_2^3 = p_4(\mathbf{t}) + y_{\mathbf{t}}$; second, we obtain $B_{\mathbf{0}_4,2} = (1/3)(p_4(\mathbf{t}) + y_{\mathbf{t}}) + (1/8)p_2^2(\mathbf{t})$.

Since $S_4(u_0, \mathbf{u}_4) \equiv 0$, we have

- (use $B_{\mathbf{0}_4,1}$) $\sum_{\mathbf{t}\in T}(-1/4)p_4(\mathbf{t})\alpha_{\mathbf{t}} = 0$;
- (use $B_{\mathbf{0}_4,2}$) $\sum_{\mathbf{t}\in T} \big((1/3)(p_4(\mathbf{t}) + y_{\mathbf{t}}) + (1/8)p_2^2(\mathbf{t})\big)\alpha_{\mathbf{t}} = 0$, which due to the previous equation is reduced to $(1/3)\sum_{\mathbf{t}\in T} y_{\mathbf{t}}\alpha_{\mathbf{t}} + (1/8)\sum_{\mathbf{t}\in T} p_2^2(\mathbf{t})\alpha_{\mathbf{t}} = 0$;
- (use $B_{\mathbf{0}_4,3}$) $\sum_{\mathbf{t}\in T} \big((-1/4)p_2^2(\mathbf{t})-(1/2)y_{\mathbf{t}}\big)\alpha_{\mathbf{t}} = (-1/4)\sum_{\mathbf{t}\in T} p_2^2(\mathbf{t})\alpha_{\mathbf{t}}-(1/2)\sum_{\mathbf{t}\in T} y_{\mathbf{t}}\alpha_{\mathbf{t}} = 0$.

Now, denote $\sum_{\mathbf{t}\in T} p_2^2(\mathbf{t})\alpha_{\mathbf{t}}$ as X and $\sum_{\mathbf{t}\in T} y_{\mathbf{t}}\alpha_{\mathbf{t}}$ as Y. From the equations above we obtain the following homogeneous linear system w.r.t. X, Y:

$$\begin{aligned} (1/8)X \quad &+(1/3)Y = 0 \\ (-1/4)X \quad &-(1/2)Y = 0 \end{aligned}$$

which has only zero solution $X = Y = 0$. Thus, we obtain $\sum_{\mathbf{t}\in T} p_2^2(\mathbf{t})\alpha_{\mathbf{t}} = 0$ from what follows that $S_6(u_0, \mathbf{u}_6) = S_6(u_0, \mathbf{0}_6)$. □

4. Example of difference equation solved by a polynomial of any degree (by any $(x-a)\dots\big(x-a-(n-1)\big)$)

This section consists of three parts. In the first part we give a difference equation such that it is solvable by any *factorial power* $(x-a)(x-a-1)\dots\big(x-a-(n-1)\big)$. In the second part we explain how we have constructed this equation, following an approach proposed in paper (van den Essen, 1992) to construct a differential equation for which any monomial x^n is a solution. In the third part, we simplify the equation to obtain the equation (2).

4.1. The equation

Let $\Delta(p)(x)$ and $\Delta^{(2)}(p)(x)$ denote differential operators $p(x)-p(x-1)$ and $\Delta(p)(x)-\Delta(p)(x-1)$ respectively. Let $H(x) := p(x)\cdot\Delta^{(2)}(p)(x) - \Delta^2(p)(x)$. It is a routine to show that the following lemma holds.

Lemma 12. Any Newton basis polynomial $g_n(x) := (x-1)\dots(x-n)$ solve the equation, w.r.t. p,

$$H(x-1)H(x) + \Delta(p)(x-1)\cdot p(x)\cdot H(x-1) - \Delta(p)(x-2)\cdot p(x-1)\cdot H(x) = 0 \tag{17}$$

Proof. First, we compute $\Delta(p)(x), \Delta^{(2)}(p)(x)$ for $p = g_n(x)$:

$$\begin{aligned}
\Delta(g_n)(x) = \; & (x-1)(x-2)\dots(x-n) - (x-2)\dots(x-n)(x-n-1) = \\
& (x-2)\dots(x-n)(x-1-x+n+1) = \\
& n(x-2)\dots(x-n) \\
\Delta^{(2)}(g_n)(x) = \; & n(x-2)(x-3)\dots(x-n) - n(x-3)\dots(x-n)(x-n-1) = \\
& n(x-3)\dots(x-n)(x-2-x+n+1) = \\
& n(n-1)(x-3)\dots(x-n)
\end{aligned}$$

Second, compute $H(x) := g_n(x) \cdot \Delta^{(2)}(g_n)(x) - \Delta^2(g_n)(x)$:

$$\begin{aligned}
H(x) & = (x-1)\dots(x-n) \cdot n(n-1)(x-3)\dots(x-n) - \\
& n^2(x-2)^2\dots(x-n)^2 = \\
& n(x-2)(x-3)^2\dots(x-n)^2\big((n-1)(x-1) - n(x-2)\big) = \\
& -n(x-2)(x-3)^2\dots(x-n)^2(x-n-1)
\end{aligned}$$

Third, compute all three summands in the l.h.s. of the equation:

$$\begin{aligned}
H_1(x) & := H(x-1)H(x) = \\
& -n(x-3)(x-4)^2\dots(x-n)^2(x-n-1)^2(x-n-2) \cdot \\
& -n(x-2)(x-3)^2\dots(x-n)^2(x-n-1) = \\
& n^2(x-2)(x-3)^3(x-4)^4\dots(x-n)^4(x-n-1)^3(x-n-2)
\end{aligned}$$

$$\begin{aligned}
H_2(x) & := \Delta(g_n)(x-1) \cdot g_n(x) \cdot H(x-1) = \\
& n(x-3)(x-4)\dots(x-n)(x-n-1) \cdot \\
& (x-1)(x-2)(x-3)(x-4)\dots(x-n) \cdot \\
& -n(x-3)(x-4)^2\dots(x-n)^2(x-n-1)^2(x-n-2) = \\
& -n^2(x-1)(x-2)(x-3)^3(x-4)^4\dots(x-n)^4(x-n-1)^3(x-n-2)
\end{aligned}$$

$$\begin{aligned}
H_3(x) & := \Delta(g_n)(x-2) \cdot g_n(x-1) \cdot H(x) = \\
& n(x-4)\dots(x-n)(x-n-1)(x-n-2) \cdot \\
& (x-2)(x-3)(x-4)\dots(x-n)(x-n-1) \cdot \\
& -n(x-2)(x-3)^2(x-4)^2\dots(x-n)^2(x-n-1) = \\
& -n^2(x-2)^2(x-3)^3(x-4)^4\dots(x-n)^4(x-n-1)^3(x-n-2)
\end{aligned}$$

Now, compute $H_1(x) + H_2(x) - H_3(x)$:

$$n^2(x-2)(x-3)^3(x-4)^4 \dots (x-n)^4(x-n-1)^3(x-n-2)\big(1-(x-1)+(x-2)\big) =$$
$$n^2(x-2)(x-3)^3(x-4)^4 \dots (x-n)^4(x-n-1)^3(x-n-2) \cdot 0 = 0$$

Therefore, direct substitution shows that any $g_n(x)$ solves equation (18). $\square$

Lemma 13. Any factorial power $(x-1)(x-a-1)\dots\big(x-a-(n-1)\big)$ solve the equation, w.r.t. p,

$$H(x-1)H(x) + \Delta(p)(x-1)\cdot p(x)\cdot H(x-1) - \Delta(p)(x-2)\cdot p(x-1)\cdot H(x) = 0 \tag{18}$$

Proof. Follows directly from Lemmata 12 and 2. $\square$

4.2. Construction

To construct the difference equation (18) we followed an approach proposed in paper (van den Essen, 1992) for a differential equation for which any monomial x^n is a solution. Construction for difference equation is similar, except that we use Newton basis polynomials, $g_n(x) = (x-1)\dots(x-n)$ for $n \geq 1$ instead of the standard monomial basis x^n. This is not surprising since $g_n(x)$ are typically considered when one speaks about topics related to difference equations.

For the sake of convenience we introduce the following shortcuts:

Definition 5.

$$\Delta_n(x) := g_n(x) - g_n(x-1)$$
$$\Delta_n^{(2)}(x) := \Delta_n(x) - \Delta_n(x-1)$$

As it was shown in Lemma (12), the following identities hold:

$$\begin{aligned}\Delta_n(x) = {} & (x-1)(x-2)\dots(x-n) - (x-2)\dots(x-n)(x-n-1) = \\ & (x-2)\dots(x-n)\big(x-1-(x-n-1)\big) = \\ & n(x-2)\dots(x-n)\end{aligned}$$

and

$$\begin{aligned}\Delta_n^{(2)}(x) = {} & n(x-2)\dots(x-n) - n(x-3)\dots(x-n)(x-n-1) = \\ & n(x-3)\dots(x-n)\big(x-2-(x-n-1)\big) = \\ & n(n-1)(x-3)\dots(x-n)\end{aligned}$$

Now,

$$
\begin{aligned}
g_n(x)\Delta_n^{(2)}(x) &= n(n-1)(x-1)(x-2)(x-3)^2\dots(x-n)^2\\
\Delta_n^2(x) &= n^2(x-2)^2(x-3)^2\dots(x-n)^2\\
g_n(x)\Delta_n^{(2)}(x)-\Delta_n^2(x) &= n(x-2)(x-3)^2\dots(x-n)^2\big((n-1)(x-1)-n(x-2)\big)=\\
&\quad n(x-2)(x-3)^2\dots(x-n)^2(nx-x-n+1-nx+2n)=\\
&\quad -n(x-2)(x-3)^2\dots(x-n)^2(x-n-1)=\\
&\quad -\Delta_n(x)\Delta_n(x-1)(1/n)
\end{aligned}
$$

Using $\dfrac{g_n(x)}{\Delta_n(x)}=\dfrac{x-1}{n}$, we obtain that

$$g_n(x)\Delta_n^{(2)}(x)-\Delta_n^2(x)=-\frac{\Delta_n(x-1)g_n(x)}{x-1}$$

and, therefore,

$$x-1=-\frac{\Delta_n(x-1)g_n(x)}{g_n(x)\Delta_n^{(2)}(x)-\Delta_n^2(x)}$$

Now we use symbolic differentiation. For all functions $h_1(x)$ and $h_2(x)$, such that $h_1(x)=h_2(x)$ it follows that $h_1(x)-h_1(x-1)=h_2(x)-h_2(x-1)$. We take $h_1(x)=x-1$ and $h_2(x)=-\dfrac{\Delta_n(x-1)g_n(x)}{g_n(x)\Delta_n^{(2)}(x)-\Delta_n^2(x)}$ and obtain

$$1=-\frac{\Delta_n(x-1)g_n(x)}{g_n(x)\Delta_n^{(2)}(x)-\Delta_n^2(x)}+\frac{\Delta_n(x-2)g_n(x-1)}{g_n(x-1)\Delta_n^{(2)}(x-1)-\Delta_n^2(x-1)}$$

By standard transformations of the fractions we obtain

$$
\begin{aligned}
&\big(g_n(x-1)\Delta_n^{(2)}(x-1)-\Delta_n^2(x-1)\big)\big(g_n(x)\Delta_n^{(2)}(x)-\Delta_n^2(x)\big)+\\
&\Delta_n(x-1)g_n(x)\big(g_n(x-1)\Delta_n^{(2)}(x-1)-\Delta_n^2(x-1)\big)-\\
&\Delta_n(x-2)g_n(x-1)\big(g_n(x)\Delta_n^{(2)}(x)-\Delta_n^2(x)\big)=0
\end{aligned}
$$

4.3. The example in the context of our settings

After substitution of Δ and $\Delta^{(2)}$ in equation (18) by their definition and simplifications, equation (18) looks as follows:

$$
\begin{aligned}
&p(x-1)p(x-3)p(x)p(x-2)-2p(x-1)^3p(x-3)+p(x-2)^2p(x-1)^2+\\
&p(x)p(x-1)^2p(x-3)-2p(x)p(x-1)p(x-2)^2+p(x-2)p(x-1)^3 \qquad =0
\end{aligned}
\tag{19}
$$

It is easy to see, that the equation (19) is equivalent to

$$
\begin{aligned}
&p(x-3)p(x)p(x-2)-2p(x-1)^2p(x-3)+p(x-2)^2p(x-1)+\\
&p(x)p(x-1)p(x-3)-2p(x)p(x-2)^2+p(x-2)p(x-1)^2 \qquad =0
\end{aligned}
\tag{20}
$$

Indeed, if equation 20 is obtained from the previous one by division by $p(x-1)$, and $p(x) \equiv 0$ is not lost as a possible solution, because it solves equation 20 as well.

If one substitutes $g_n(x) = \prod_{i=1}^{n}(x-i)$ into the l.h.s. of the equation then one directly sees that after evaluation the result of substitution is equal to zero. In terms of Definition 2 it means that for all $n \geq 1$ for all $l \geq 0$ one has

$$S\big(n, p_1(\mathbf{n}), \dots, p_\ell(\mathbf{n})\big) = 0, \text{ where } \mathbf{n} = (1, \dots, n)$$

It would be an interesting exercise to check if every $S_\ell\big(n, 0, \dots, 0\big) = 0$ for all n (that is $S_\ell\big(n, 0, \dots, 0\big)$ is the constant zero).

5. Difference equations with a single shift

Consider difference equations of the form

$$G(P(x), P(x-\tau)) + G_0(x) = 0. \tag{21}$$

In this equation $\tau_1 = 0$, $\tau_2 = \tau$. For the sake of convenience denote $(0, \dots, 0, \tau, \dots, \tau)$, where τ occurs m times, by $\mathbf{t}(m)$. The aim is to prove

Theorem 6. *The degree of a polynomial solution P of equation (21), if it exists, is $d \leq \max\{D, \deg(G_0)/(D-1)\}$.*

To show that the conditions of the framework lemma are satisfied consider a few facts about p_ℓ and $S_\ell(u_0, \mathbf{u}_\ell)$ for equation (21). First of all, it is easy to see that $p_\ell\big(\mathbf{t}(m)\big) = 0^\ell + \ldots + 0^\ell + \tau^\ell + \ldots + \tau^\ell = m\tau^\ell$. Second, from this and identity (6) it follows that $p_\ell\big(\mathbf{t}(m) + \mathbf{r}\big) = m \sum_{\lambda=0}^{\ell} \binom{\ell}{\lambda} \tau^{\ell-\lambda} p_\lambda(\mathbf{r})$, where $\ell \geq 1$. Third, by the definition of E_ℓ for $\ell \geq 1$ one obtains

$$E_\ell(D, \mathbf{p}_\ell\big(\mathbf{t}(m)\big), u_0, \mathbf{u}_\ell) = -(m/\ell) \textstyle\sum_{\kappa=1}^{\ell} E_{\ell-\kappa}(D, \mathbf{p}_{\ell-\kappa}\big(\mathbf{t}(m)\big), u_0, \mathbf{u}_{\ell-\kappa}) \sum_{\lambda=0}^{\kappa} \binom{\kappa}{\lambda} \tau^{k-\lambda} u_\lambda. \tag{22}$$

From this one obtains the following recurrent formulae:

$$\begin{aligned}
A_{\mathbf{i}_\ell}\Big(D, \mathbf{p}_\ell\big(\mathbf{t}(m)\big), u_0\Big) &= (-m/\ell)\Big(\textstyle\sum_{\kappa=1}^{\ell} \sum_{\lambda=1}^{\kappa} \binom{\kappa}{\lambda} \tau^{\kappa-\lambda} A_{\mathbf{i}_{\ell-\kappa}-1_\lambda}\Big(D, \mathbf{p}_{\ell-\kappa}\big(\mathbf{t}(m)\big), u_0\Big) + \\
&\quad \textstyle\sum_{\kappa=1}^{\ell} u_0\, \tau^\kappa A_{\mathbf{i}_{\ell-\kappa}}\Big(D, \mathbf{p}_{\ell-\kappa}\big(\mathbf{t}(m)\big), u_0\Big)\Big) \\
&\quad \text{where } \mathbf{i}_{\ell-\kappa} - 1_\lambda := (i_1, \dots, i_{\lambda-1}, i_\lambda - 1, i_{\lambda+1}, \dots, i_{\ell-\kappa}) \\
&\quad \text{and } A_{\mathbf{i}_{\ell-\kappa}-1_\lambda} = 0 \text{ if } i_\lambda = 0, \text{ or } \lambda > \ell - \kappa \\
B_{\mathbf{i}_\ell,\mu}\Big(D, \mathbf{p}_\ell\big(\mathbf{t}(m)\big)\Big) &= (-m/\ell)\Big(\textstyle\sum_{\kappa=1}^{\ell} \sum_{\lambda=1}^{\kappa} \binom{\kappa}{\lambda} \tau^{\kappa-\lambda} B_{\mathbf{i}_{\ell-\kappa}-1_\lambda,\mu}\Big(D, \mathbf{p}_{\ell-\kappa}\big(\mathbf{t}(m)\big)\Big) + \\
&\quad \textstyle\sum_{\kappa=1}^{\ell} \tau^\kappa B_{\mathbf{i}_{\ell-\kappa},\mu-1}\Big(D, \mathbf{p}_{\ell-\kappa}\big(\mathbf{t}(m)\big)\Big)\Big).
\end{aligned} \tag{23}$$

One refines the upper limits on κ of the sums in the identities for $B_{\mathbf{i}_\ell,\mu}$ in the following way. From the definition of $B_{\mathbf{i}_\ell,\mu}$ in the first sum one has $0 \leq \mu \leq \ell - \kappa$, which means that $\kappa \leq \ell - \mu$. For the second sum one has $\kappa \leq \ell - \mu + 1$ because $0 \leq \mu - 1 \leq \ell - \kappa$.

Lemma 14. For any $\ell \geq 1$ the following statement holds: for all $1 \leq \mu \leq \ell$ there exists a constant $C_{\ell,\mu} > 0$ such that

$$B_{\mathbf{0}_\ell,\mu}\Big(D, \mathbf{p}_\ell\big(\mathbf{t}(m)\big)\Big) = (-1)^\mu C_{\ell,\mu} m^\mu \tau^\ell. \tag{24}$$

Proof. We prove the lemma by induction on ℓ. To begin with, for $\ell = 1$ one has $B_{\mathbf{0}_1,1}(v_0, v_1) = -v_1$. Therefore $B_{\mathbf{0}_1,1}\Big(D, p_1\big(\mathbf{t}(m)\big)\Big) = -m\tau = (-1)^1 m^1 \tau^1$, so $C_{1,1} = 1$.

Now, fix some $\ell > 1$. Use the recurrent formula for $B_{\mathbf{0}_\ell,\mu}$:

$$B_{\mathbf{0}_\ell,\mu}\Big(D, \mathbf{p}_\ell\big(\mathbf{t}(m)\big)\Big) = -(m/\ell) \textstyle\sum_{\kappa=1}^{\ell-\mu+1} \tau^\kappa B_{\mathbf{0}_{\ell-\kappa},\mu-1}\Big(D, \mathbf{p}_{\ell-\kappa}\big(\mathbf{t}(m)\big)\Big). \tag{25}$$

By the induction assumption the statement of the lemma holds for all $\ell' < \ell$. This implies that $B_{\mathbf{0}_\ell,\mu}\Big(D, \mathbf{p}_\ell\big(\mathbf{t}(m)\big)\Big)$ is equal to

$$-(m/\ell) \sum_{\kappa=1}^{\ell-\mu+1} \tau^\kappa \tau^{\ell-\kappa} (-1)^{\mu-1} m^{\mu-1} C_{\ell-\kappa,\mu-1} = \tau^\ell m^{1+\mu-1} (-1)^{1+\mu-1} (1/\ell) \sum_{\kappa=1}^{\ell-\mu+1} C_{\ell-\kappa,\mu-1}. \tag{26}$$

From this it follows that $C_{\ell,\mu} = (1/\ell) \sum_{\kappa=1}^{\ell-\mu+1} C_{\ell-\kappa,\mu-1} > 0$. $\square$

Now one can prove Theorem 6.

Proof. We show that the conditions of the framework lemma hold. To begin with, we show that there exists $0 \leq L \leq D$ such that the polynomial $S_L(u_0, \mathbf{0}_L)$ is not everywhere zero. Assume the opposite: $S_\ell(u_0, \mathbf{0}_\ell)$ are everywhere zero for all $0 \leq \ell \leq D$. This implies that the corresponding coefficients of u_0^μ in $S_\ell(u_0, \mathbf{0}_\ell)$ must be all zeros. Hence by Lemma 14 with $\mu := \ell$ it follows that $\sum_{m=0}^{D} (-1)^\ell C_{\ell,\ell} \tau^\ell m^\ell \alpha_{\mathbf{t}(m)} = 0$ which due to $\tau \neq 0$ and $C_{\ell,\ell} > 0$ implies $\sum_{m=0}^{D} m^\ell \alpha_{\mathbf{t}(m)} = 0$ for all $0 \leq \ell \leq D$. That is, one gets a system of $D+1$ linear equations for $D+1$ variables x_m. The matrix of this system is of rank $D+1$ because its determinant is equal to the $D \times D$ Vandermonde determinant. Therefore, the system has only the zero solution $\alpha_{\mathbf{t}(m)}$ which contradicts the fact that G is of degree D. Therefore there exists $S_L(u_0, \mathbf{0}_L)$ which is not everywhere zero. W.l.o.g. assume that for all $0 \leq \ell \leq L-1$ the polynomials $S_\ell(u_0, \mathbf{0}_\ell)$ are everywhere zero.

Note that if $L = 0$ then $S_0(u_0) \neq 0$. Comparing the left- and right-hand sides of the corresponding equation of the form (3) yields $dD \leq \max\{\deg(G_0), d(D-1)\}$. Therefore, $d \leq \deg(G_0)/D \leq \deg(G_0)/(D-1) \leq \max\{D, \deg(G_0)/(D-1)\}$.

Now we consider the case $L \geq 1$ in more detail. The function $A_{\mathbf{i}_\ell}\Big(D, \mathbf{p}_\ell\big(\mathbf{t}(m)\big), u_0\Big)$ can be seen as a polynomial in m because $\mathbf{p}_\ell\big(\mathbf{t}(m)\big) = m\tau^\ell$. Let $T^{D,\tau}_{\mathbf{i}_\ell,\mu}(u_0)$ denote its coefficients of m^μ. Since $A_{\mathbf{i}_\ell}\Big(D, \mathbf{p}_\ell\big(\mathbf{t}(m)\big), u_0\Big)$ is a linear combination of m^μ, it is a linear combination of $B_{\mathbf{0}_\mu,\mu}\Big(D, \mathbf{p}_\mu\big(\mathbf{t}(m)\big)\Big) = (-1)^\mu m^\mu C_{\mu,\mu} \tau^\mu$ as well, with the coefficients $H^\tau_{\mathbf{i}_\ell,\mu,\mu}(D, u_0) = (-1)^\mu T^{D,\tau}_{\mathbf{i}_\ell,\mu}(u_0)/(C_{\mu,\mu}\tau^\mu)$.

Assume that $d > \max\{L, \deg(G_0)/(D-1)\} \geq 0$. By the framework lemma one obtains that d is a root of $S_L(u_0, \mathbf{0}_L)$. Since $S_L(u_0, \mathbf{0}_L) = \sum_{m=0}^{D} \alpha_{\mathbf{t}(m)} A_{\mathbf{0}_L}\Big(D, \mathbf{p}_L\big(\mathbf{t}(m)\big), u_0\Big) =$

$\sum_{\mu=0}^{L} u_0^{\mu} \sum_{m=0}^{D} \alpha_{\mathbf{t}(m)} B_{\mathbf{0}_L,\mu}\Big(D, \mathbf{p}_L\big(\mathbf{t}(m)\big)\Big)$, by Lemma 14 one obtains that $S_L(u_0, \mathbf{0}_L) = \sum_{\mu=0}^{L} u_0^{\mu}(-1)^{\mu}\tau^L C_{L,\mu} \sum_{m=0}^{D} m^{\mu}\alpha_{\mathbf{t}(m)}$. Consider the sums $\sum_{m=0}^{D} m^{\mu}\alpha_{\mathbf{t}(m)}$ for $0 \le \mu \le L-1$. Since the polynomials $S_\mu(u_0, \mathbf{0}_\mu)$ are everywhere zero for all $0 \le \mu \le L-1$ one gets $\sum_{m=0}^{D}(-1)^{\mu} C_{\mu,\mu}\tau^{\mu} m^{\mu}\alpha_{\mathbf{t}(m)} = 0$, which implies that $\sum_{m=0}^{D} m^{\mu}\alpha_{\mathbf{t}(m)} = 0$ for $0 \le \mu \le L-1$. Therefore, $S_L(u_0, \mathbf{0}_L) = u_0^L(-1)^L\tau^L C_{L,L}\sum_{m=0}^{D} m^L\alpha_{\mathbf{t}(m)} = 0$. Since $S_L(u_0, \mathbf{0}_L)$ is not everywhere zero in u_0 the inequation $\sum_{m=0}^{D} m^L \alpha_{\mathbf{t}(m)} \neq 0$ holds. From this it follows that $S_L(d, \mathbf{0}_L) = 0$ implies $d = 0$, which contradicts the assumption $d > 0$.

Therefore, $d \le \max\{L, \deg(G_0)/(D-1)\} \le \max\{D, \deg(G_0)/(D-1)\}$. □

6. Example

The equation $P(x) = P^2(x-1) - 2P(x-1)P(x-2) + 3P(x-1)P(x-3) - 2P^2(x-2) - 17P(x-1) + 29x^2 - 45x + 51$ has a polynomial solution of degree $d = 3$ which is a root of the degree polynomial. Here $D = 2$ and $\deg(G_0)/(D-1) = 2$. To find the degree polynomial it is enough to calculate $S_0(u_0)$, $S_1(u_0, 0)$ and $S_2(u_0, 0, 0)$. Use the definition

$$S_\ell(u_0, \mathbf{0}_\ell) = \sum_{\mathbf{t}\in T} A_{\mathbf{0}_\ell}\big(D, \mathbf{p}_\ell(\mathbf{t}), u_0\big)\alpha_{\mathbf{t}}.$$

Direct calculation yields $A_{()}\big(v_0, (), u_0, ()\big) = 1$, $A_{(0)}\big(v_0, (v_1), u_0\big) = -v_1 u_0$ and $A_{(0,0)}(v_0, \mathbf{v}_2, u_0) = (1/2)v_1^2u_0^2 - (1/2)v_2u_0$ (see the Appendix, Section 8.1). Compute the values $p_\ell(\mathbf{t})$ (for non-vanishing $\alpha_{\mathbf{t}}$):

$\mathbf{t}$	$p_1(\mathbf{t})$	$p_1^2(\mathbf{t})$	$p_2(\mathbf{t})$	$\alpha_{\mathbf{t}}$
$(1,1)$	$1+1=2$	4	$1^2+1^2=2$	1
$(1,2)$	$1+2=3$	9	$1^2+2^2=5$	-2
$(1,3)$	$1+3=4$	16	$1^2+3^2=10$	3
$(2,2)$	$2+2=4$	16	$2^2+2^2=8$	-2

As one can see from the equation, the coefficients $\alpha_{\mathbf{t}}$ for $\mathbf{t}$ that are not mentioned in the table vanish. Now, by the substitutions $v_\ell := p_\ell(\mathbf{t})$ one obtains

$$\begin{aligned}
S_0(u_0) &= \textstyle\sum_{\mathbf{t}\in T}\alpha_{\mathbf{t}} = 1 - 2 + 3 - 2 = 0 \text{ for all values of } u_0\\
S_1(u_0, 0) &= u_0\textstyle\sum_{\mathbf{t}\in T} p_1(\mathbf{t})\alpha_{\mathbf{t}} = u_0(1\cdot 2 - 2\cdot 3 + 3\cdot 4 - 2\cdot 4) = 0 \text{ for all values of } u_0\\
S_2(u_0, 0, 0) &= u_0(1/2)\Big(\textstyle\sum_{\mathbf{t}\in T}\big(u_0p_1^2(\mathbf{t}) - p_2(\mathbf{t})\big)\alpha_{\mathbf{t}}\Big) =\\
&\quad (u_0/2)\big(u_0\cdot(1\cdot 4 - 2\cdot 9 + 3\cdot 16 - 2\cdot 16) -\\
&\quad (1\cdot 2 - 2\cdot 5 + 3\cdot 10 - 2\cdot 8)\big) = u_0(u_0 - 3).
\end{aligned}$$

So, here $L = 2$. From this it follows, that if the difference equation has a polynomial solution of degree $d > L$ then for this degree it must hold $d = 3$. It is easy to check that there is a solution $P(x) = x^3 + x^2 + x + 1$ for the equation.

7. Conclusions and outlook

The present article is concerned with polynomial solutions $P(x)$ of difference equations of the form $G(P(x-\tau_1), \ldots, P(x-\tau_s)) + G_0(x) = 0$, where $G(x_1, \ldots, x_s)$ is a known polynomial of degree $D \geq 2$ and G_0 is a known polynomial in x. The authors address the cases when one can bound the degree d of a polynomial solution P if such a solution exists. For the difference equation a family of polynomials $S_\ell(u_0, \mathbf{0}_\ell)$, $\ell \geq 0$, has been defined, and it has been shown that if $\mathcal{L} := \{\ell | S_\ell(u_0, \mathbf{0}_\ell) \text{ is not everywhere zero}\} \neq \emptyset$ and $L := \min(\mathcal{L}) \leq 5$ then $d \leq \max\{L, \deg(G_0)/(D-1)\}$ or d must be among the positive integer roots of $S_L(u_0, \mathbf{0}_L)$ (Theorem 4). Also, it has been shown that in this way one can bound d for all quadratic difference equations with $\tau_i = a + i - 1$, where $i = 1, 2, 3$, and all cubic difference equations with $\tau_i = a + i - 1$ where $i = 1, 2$. In general, with the presented approach it is impossible to bound the degree of solutions of difference equations for which $S_\ell(u_0, \mathbf{0}_\ell)$ are everywhere zero for all $0 \leq \ell \leq 5$. However, it has been proven that $d \leq \max\{D, \deg(G_0)/(D-1)\}$ for equations with $s = 2$, $\tau_1 = 0$ and $\tau_2 = \tau$, see Theorem 6.

An obvious direction of future research is applying the presented technique to polynomial difference equations with polynomial non-constant coefficients. A more challenging problem is to check if there are connections between the obtained results and Galois theory.

From the application point of view the obtained result improves polynomial resource analysis of computer programs developed in article (Shkaravska et al., 2009). There the authors consider the size of an output as a polynomial function on the sizes of inputs. In the Charter project the authors developed the ResAna tool (Kersten et al., 2012) that applies polynomial interpolation to generate an upper bound on Java loop iterations. The tool requires the user to input the degree of the solution. The results of this article will help to automatically obtain the degree of the polynomial in many cases.

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8. Appendix

This appendix assists the proofs of Lemmata 10 and 11, and Corollary 1. In Section 8.1 the coefficients $B_{\mathbf{i}_\ell,\mu}(v_0,\mathbf{v}_\ell)$ and $H_{\mathbf{i}_L,\ell,\mu}(v_0,u_0)$ are listed. They are referred to in the proof of Lemmata 10 and 11. In Section 8.2 one finds the matrix of the linear system for $\bar{\alpha}$ which is used in the proof of Corollary 1.

8.1. The coefficients $B_{\mathbf{i}_\ell,\mu}(v_0,\mathbf{v}_\ell)$ and $H_{\mathbf{i}_L,\ell,\mu}(v_0,u_0)$.

In this section we consider the polynomials $B_{\mathbf{i}_\ell,\mu}(v_0,\mathbf{v}_\ell)$ which are the coefficients of the monomials u_0^μ of $A_{\mathbf{i}_\ell}(v_0,\mathbf{v}_\ell,u_0)$. Moreover, we give the coefficients $H_{\mathbf{i}_L,\ell,\mu}(v_0,u_0)$ for the representations of $A_{\mathbf{i}_L}(v_0,\mathbf{v}_\ell,u_0)$ as the linear combinations of $B_{\mathbf{0}_\ell,\mu}(v_0,\mathbf{v}_\ell)$, for $\mathbf{i}_L \neq \mathbf{0}_L$. Calculations are performed in `lemma-7.mw` and `lemma-8.mw`.

- ($L = 0$.) In this case $E_0\big(v_0,(),u_0,()\big) = 1$ and one gets $B_{(),0} = A_{()} = 1$ immediately by the definitions.
- ($L = 1$.) In this case $E_1\big(v_0,\mathbf{v}_1,u_0,\mathbf{u}_1\big) = -v_0u_1 - v_1u_0$, and therefore $A_{(0)}(v_0,\mathbf{v}_1,u_0) = -u_0v_1$ and $A_{(1)}(v_0,\mathbf{v}_1,u_0) = -v_0$. From this follows that $B_{(0),1}(v_0,v_1) = -v_1$ and $B_{(0),0}(v_0,v_1) = 0$. At the end of the day one obtains $H_{(1),0,0} = -v_0$, since $A_{(1)}(v_0,v_1,u_0) = -v_0B_{(),0}$.
- ($L = 2$.) Then $E_2(v_0,\mathbf{v}_2,u_0,\mathbf{u}_2) = (1/2)u_0^2v_1^2 + (1/2)v_0^2u_1^2 + v_1u_0v_0u_1 - (1/2)v_2u_0 - v_1u_1 - (1/2)v_0u_2$ and $A_{(0,0)}(v_0,\mathbf{v}_2,u_0) = (1/2)v_1^2u_0^2 - (1/2)v_2u_0$. From this follows that

$B_{(0,0),2}(v_0,\mathbf{v}_2) = (1/2)v_1^2$	$B_{(0,0),1}(v_0,\mathbf{v}_2) = -(1/2)v_2$	$B_{(0,0),0}(v_0,\mathbf{v}_2) = 0$

and

$H_{(1,0),1,1} = (-u_0v_0 + 1)$	$H_{(2,0),0,0} = (1/2)v_0^2$	$H_{(0,1),0,0} = -(1/2)v_0$

- ($L = 3$.) Then for $E_3(v_0, \mathbf{v}_3, u_0, \mathbf{u}_3)$ one obtains $A_{(0,0,0)} = -(1/6)v_1^3u_0^3 - (1/3)v_3u_0 + (1/2)v_2u_0^2v_1$. From this follows that $A_{(0,0,0)} = -(1/6)v_1^3u_0^3 - (1/3)v_3u_0 + (1/2)v_2u_0^2v_1$ and

$B_{(0,0,0),3}(v_0, \mathbf{v}_3) = -(1/6)v_1^3$	$B_{(0,0,0),2}(v_0, \mathbf{v}_3) = (1/2)v_1v_2$	$B_{(0,0,0),1}(v_0, \mathbf{v}_3) = -(1/3)v_3$
$B_{(0,0,0),0}(v_0, \mathbf{v}_3) = 0$		

Eventually,

$H_{(1,0,0),2,2} = -u_0^2v_0 + 2u_0$	$H_{(1,0,0),2,1} = -u_0v_0 + 2$	$H_{(2,0,0),1,1} = (1/2)v_0^2u_0 - v_0$
$H_{(3,0,0),0,0} = -(1/6)v_0^3$	$H_{(0,1,0),1,1} = -(1/2)u_0v_0 + 1$	$H_{(1,1,0),0,0} = (1/2)v_0^2$
$H_{(0,0,1),0,0} = -(1/3)v_0$		

and the other $H_{\mathbf{i}_3,\ell,\mu}$ vanish.

- ($L = 4$) For $E_4(v_0, \mathbf{v}_4, u_0, \mathbf{u}_4)$ symbolic computation yields that

$B_{(0,0,0,0),4} = (1/24)v_1^4$	$B_{(0,0,0,0),3} = -(1/4)v_2v_1^2$	$B_{(0,0,0,0),2} = (1/3)v_3v_1 + (1/8)v_2^2$
$B_{(0,0,0,0),1} = -(1/4)v_4$	$B_{(0,0,0,0),0} = 0$	

and

$H_{(1,0,0,0),3,3} = -v_0u_0^3 + 3u_0^2$	$H_{(1,0,0,0),3,2} = -v_0u_0^2 + 3u_0$	$H_{(1,0,0,0),3,1} = -v_0u_0 + 3$
$H_{(2,0,0,0),2,2} =$ $(1/2)v_0^2u_0^2 - 2v_0u_0 + 1$	$H_{(2,0,0,0),2,1} = (1/2)v_0^2u_0 - 2v_0$	$H_{(3,0,0,0),1,1} =$ $-(1/6)v_0^3u_0 + (1/2)v_0^2$
$H_{(4,0,0,0),0,0} = (1/24)v_0^4$	$H_{(0,1,0,0),2,2} = -(1/2)v_0u_0^2 + 2u_0$	$H_{(0,1,0,0),2,1} =$ $-(1/2)v_0u_0 + 3$
$H_{(0,2,0,0),0,0} = (1/8)v_0^2$	$H_{(0,0,1,0),1,1} = -(1/3)v_0u_0 + 1$	$H_{(0,0,0,1),0,0} = -(1/4)v_0$
$H_{(1,1,0,0),1,1} =$ $(1/2)v_0^2u_0 - (3/2)v_0$	$H_{(2,1,0,0),0,0} = -(1/4)v_0^3$	$H_{(1,0,1,0),0,0} = (1/3)v_0^2$

The other $H_{\mathbf{i}_3,\ell,\mu}$ vanish.

- ($L = 5$.) Symbolic computation of $E_5(v_0, \mathbf{v}_5, u_0, \mathbf{u}_5)$ gives

$B_{\mathbf{0}_5,5}(v_0, \mathbf{v}_5) = -(1/120)v_1^5$	$B_{\mathbf{0}_5,4} = (1/12)v_2v_1^3$	$B_{\mathbf{0}_5,3} = -(1/6)v_1^2v_3 - (1/8)v_2^2v_1$
$B_{\mathbf{0}_5,2} = (1/4)v_4v_1 + (1/6)v_3v_2$	$B_{\mathbf{0}_5,1} = -(1/5)v_5$	$B_{\mathbf{0}_5,0} = 0$

Now we list the coefficients of $A_{\mathbf{i}_5}(v_0, \mathbf{v}_5, u_0)$ considered as linear combinations of $B_{\mathbf{0}_\ell,\mu}(v_0, \mathbf{v}_\ell)$ where $\ell \leq 4$:

$H_{(1,0,0,0,0),4,4} = -v_0 u_0^4 + 4u_0^3$	$H_{(1,0,0,0,0),4,3} = 4u_0^2 - v_0 u_0^3$
$H_{(1,0,0,0,0),4,2} = -v_0 u_0^2 + 4u_0$	$H_{(1,0,0,0,0),4,1} = -u_0 v_0 + 4$
$H_{(2,0,0,0,0),3,3} = (1/2)v_0^2 u_0^3 - 3v_0 u_0^2 + 3u_0$	$H_{(2,0,0,0,0),3,2} = (1/2)v_0^2 u_0^2 - 3u_0 v_0 + 2$
$H_{(2,0,0,0,0),3,1} = -3v_0 + (1/2)v_0^2 u_0$	$H_{(3,0,0,0,0),2,2} = -(1/6)v_0^3 u_0^2 + v_0^2 u_0 - v_0$
$H_{(3,0,0,0,0),2,1} = -(1/6)v_0^3 u_0 + v_0^2$	$H_{(4,0,0,0,0),1,1} = (1/24)v_0^4 u_0 - (1/6)v_0^3$
$H_{(5,0,0,0,0),0,0} = -(1/120)v_0^5$	$H_{(0,1,0,0,0),3,3} = -(1/2)v_0 u_0^3 + 3u_0^2$
$H_{(0,1,0,0,0),3,2} = -(1/2)v_0 u_0^2 + 4u_0$	$H_{(0,1,0,0,0),3,1} = -(1/2)u_0 v_0 + 6$
$H_{(0,2,0,0,0),1,1} = (1/8)v_0^2 u_0 - (1/2)v_0$	$H_{(1,1,0,0,0),2,2} = (1/2)v_0^2 u_0^2 - 3v_0 u_0 + 2$
$H_{(1,1,0,0,0),2,1} = (1/2)v_0^2 u_0 - 4v_0$	$H_{(1,2,0,0,0),1,1} = -(1/8)v_0^3$
$H_{(0,0,1,0,0),2,2} = -(1/3)v_0 u_0^2 + 2u_0$	$H_{(0,0,1,0,0),2,1} = -(1/3)u_0 v_0 + 4$
$H_{(1,0,1,0,0),1,1} = (1/3)v_0^2 u_0 - (4/3)v_0$	$H_{(2,1,0,0,0),1,1} = -(1/4)v_0^3 u_0 + v_0^2$
$H_{(2,0,1,0,0),0,0} = -(1/6)v_0^3$	$H_{(3,1,0,0,0),0,0} = (1/12)v_0^4$
$H_{(0,1,1,0,0),0,0} = (1/6)v_0^2$	$H_{(0,0,0,1,0),1,1} = -(1/4)u_0 v_0 + 1$
$H_{(1,0,0,1,0),0,0} = (1/4)v_0^2$	$H_{(0,0,0,0,1),0,0} = -(1/5)v_0$

The coefficients that are not in the list vanish.

See the next page for $L = 6$.

- ($L = 6$.) The representation of $A_{\mathbf{i}_6}(v_0, \mathbf{v}_6, u_0)$ is considered in detail in the tables below.

$\mathbf{i}_6$	$H_{\mathbf{i}_6,5,5}$	$H_{\mathbf{i}_6,5,4}$	$H_{\mathbf{i}_6,5,3}$	$H_{\mathbf{i}_6,5,2}$	$H_{\mathbf{i}_6,5,1}$
100000	$-v_0u_0^5 + 5u_0^4$	$5u_0^3 - v_0u_0^4$	$-v_0u_0^3 + 5u_0^2$	$-u_0^2v_0 + 5u_0$	$-u_0v_0 + 5$
200000	0	0	0	0	0
300000	0	0	0	0	0
400000	0	0	0	0	0
500000	0	0	0	0	0
600000	0	0	0	0	0
010000	0	0	0	0	0
020000	0	0	0	0	0
030000	0	0	0	0	0
110000	0	0	0	0	0
210000	0	0	0	0	0
310000	0	0	0	0	0
410000	0	0	0	0	0
120000	0	0	0	0	0
220000	0	0	0	0	0
001000	0	0	0	0	0
002000	0	0	0	0	0
101000	0	0	0	0	0
201000	0	0	0	0	0
301000	0	0	0	0	0
011000	0	0	0	0	0
111000	0	0	0	0	0
000100	0	0	0	0	0
100100	0	0	0	0	0
200100	0	0	0	0	0
010100	0	0	0	0	0
000010	0	0	0	0	0
100010	0	0	0	0	0
000001	0	0	0	0	0

$\mathbf{i}_6$	$H_{\mathbf{i}_6,4,4}$	$H_{\mathbf{i}_6,4,3}$	$H_{\mathbf{i}_6,4,2}$	$H_{\mathbf{i}_6,4,1}$	$H_{\mathbf{i}_6,3,3}$
100000	0	0	0	0	0
200000	$-4v_0u_0^3+ (1/2)u_0^4v_0^2+ 6u_0^2$	$-4v_0u_0^2+ 5u_0+ (1/2)u_0^3v_0^2$	$3-4v_0u_0+ (1/2)u_0^2v_0^2$	$-4v_0+ (1/2)u_0v_0^2$	0
300000	0	0	0	0	$\frac{3}{2}u_0^2v_0^2- 3v_0u_0- \frac{1}{6}u_0^3v_0^3+1$
400000	0	0	0	0	0
500000	0	0	0	0	0
600000	0	0	0	0	0
010000	$-(1/2)u_0^4v_0+ 4u_0^3$	$-(1/2)u_0^3v_0+ 5u_0^2$	$-(1/2)u_0^2v_0+ 7u_0$	$-(1/2)u_0v_0+ 10$	0
020000	0	0	0	0	0
030000	0	0	0	0	0
110000	0	0	0	0	$(1/2)u_0^3v_0^2- \frac{9}{2}u_0^2v_0+ 6u_0$
210000	0	0	0	0	0
310000	0	0	0	0	0
410000	0	0	0	0	0
120000	0	0	0	0	0
220000	0	0	0	0	0
001000	0	0	0	0	$-\frac{1}{3}u_0^3v_0+ 3u_0^2$
002000	0	0	0	0	0
101000	0	0	0	0	0
201000	0	0	0	0	0
301000	0	0	0	0	0
011000	0	0	0	0	0
111000	0	0	0	0	0
000100	0	0	0	0	0
100100	0	0	0	0	0
200100	0	0	0	0	0
010100	0	0	0	0	0
000010	0	0	0	0	0
100010	0	0	0	0	0
000001	0	0	0	0	0

	$H_{\mathbf{i}_6,3,2}$	$H_{\mathbf{i}_6,3,1}$	$H_{\mathbf{i}_6,2,2}$	$H_{\mathbf{i}_6,2,1}$	$H_{\mathbf{i}_6,1,1}$	$H_{\mathbf{i}_6,0,0}$
100000	0	0	0	0	0	0
200000	0	0	0	$-\frac{1}{4}v_2$	0	0
300000	$\frac{3}{2}v_0^2u_0-\frac{1}{6}v_0^3u_0^2-2v_0$	$-\frac{1}{6}v_0^3u_0+\frac{3}{2}v_0^2$	0	0	0	0
400000	0	0	$\frac{1}{2}v_0^2-\frac{1}{3}v_0^3u_0+\frac{1}{24}v_0^4u_0^2$	$\frac{1}{24}v_0^4u_0-\frac{1}{3}v_0^3$	0	0
500000	0	0	0	0	$\frac{1}{24}v_0^4-\frac{1}{120}v_0^5u_0$	0
600000	0	0	0	0	0	$\frac{1}{720}v_0^6$
010000	0	0	0	$\frac{1}{4}v_2u_0$	0	0
020000	0	0	$\frac{1}{8}u_0^2v_0^2-u_0v_0+1$	$\frac{1}{8}v_0^2u_0-\frac{3}{2}v_0$	0	0
030000	0	0	0	0	0	$-\frac{1}{48}v_0^3$
110000	$\frac{1}{2}v_0^2u_0^2-\frac{11}{2}u_0v_0+5$	$-\frac{15}{2}v_0+\frac{1}{2}u_0v_0^2$	0	0	0	0
210000	0	0	$2v_0^2u_0-\frac{5}{2}v_0-\frac{1}{4}v_0^3u_0^2$	$\frac{5}{2}v_0^2-\frac{1}{4}v_0^3u_0$	0	0
310000	0	0	0	0	$-\frac{5}{12}v_0^3+\frac{1}{12}v_0^4u_0$	0
410000	0	0	0	0	0	$-\frac{1}{48}v_0^5$
120000	0	0	0	0	$-\frac{1}{8}v_0^3u_0+\frac{5}{8}v_0^2$	0
220000	0	0	0	0	0	$\frac{1}{16}v_0^4$
001000	$-\frac{1}{3}u_0^2v_0+5u_0$	$-\frac{1}{3}u_0v_0+10$	0	0	0	0
002000	0	0	0	0	0	$\frac{1}{18}v_0^2$
101000	0	0	$\frac{1}{3}v_0^2u_0^2-\frac{8}{3}u_0v_0+2$	$\frac{1}{3}v_0^2u_0-\frac{14}{3}v_0$	0	0
201000	0	0	0	0	$-\frac{1}{6}u_0v_0^3+\frac{5}{6}v_0^2$	0
301000	0	0	0	0	0	$\frac{1}{18}v_0^4$
011000	0	0	0	0	$\frac{1}{6}v_0^2u_0-\frac{5}{6}v_0$	0
111000	0	0	0	0	0	$-\frac{1}{6}v_0^3$
000100	0	0	$-\frac{1}{4}u_0^2v_0+2u_0$	$-\frac{1}{4}u_0v_0+5$	0	0
100100	0	0	0	0	$\frac{1}{4}v_0^2u_0-\frac{5}{4}v_0$	0
200100	0	0	0	0	0	$-\frac{1}{8}v_0^3$
010100	0	0	0	0	0	$\frac{1}{8}v_0^2$
000010	0	0	0	0	$-\frac{1}{5}v_0+1$	0
100010	0	0	0	0	0	$\frac{1}{5}v_0^2$
000001	0	0	0	0	0	$-\frac{1}{6}v_0$

Now we give the coefficients for the representation of $A_{(0,1,0,0,0,0)}(v_0, \mathbf{v}_6, u_0)$ and $A_{(2,0,0,0,0,0)}(v_0, \mathbf{v}_6, u_0)$ via the alternative list of polynomials $B_{\mathbf{0}_\ell,\mu}(v_0, \mathbf{v}_\ell)$ where $\mu \leq \ell \leq 5$, with $B_{\mathbf{0}_2,1}(v_0, \mathbf{v}_2)$ replaced by v_2^2:

$H_{(0,1,0,0,0,0)v_2^2} = -(1/8)u_0$	$H_{(0,1,0,0,0,0),4,1} = 10 - (1/2)u_0v_0$
$H_{(0,1,0,0,0,0),4,2} = -(1/2)u_0^2v_0 + 7u_0$	$H_{(0,1,0,0,0,0),4,3} = -(1/2)u_0^3v_0 + 5u_0^2$
$H_{(0,1,0,0,0,0),4,4} = -(1/2)u_0^4v_0 + 4u_0^3$	
$H_{(2,0,0,0,0,0),v_2^2} = -(1/8)$	$H_{(2,0,0,0,0,0),4,1} = -4v_0 + (1/2)u_0v_0^2$
$H_{(2,0,0,0,0,0),4,2} = (1/2)u_0^2v_0^2 + 3 - 4u_0v_0$	$H_{(2,0,0,0,0,0),4,3} = 5u_0 + (1/2)u_0^3v_0^2 - 4u_0^2v_0$
$H_{(2,0,0,0,0,0),4,4} = (1/2)u_0^4v_0^2 + 6u_0^2 - 4u_0^3v_0$	

8.2. Difference equations $G\big(P(x), P(x-1), P(x-2)\big) + G_0(x) = 0$, $D = 2$.

In this section we give the matrix of the linear system for $\bar{\alpha}$, encountered in solving quadratic difference equations of the form $G\big(P(x), P(x-1), P(x-2)\big) + G_0(x) = 0$, see Corollary 1. The matrix is computed and the system is solved in `corollaries_Ds.mw`, which is available at `http://resourceanalysis.cs.ru.nl/`, in the archive mentioned in the Introduction.

$$\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
0 & -1 & -2 & -2 & -3 & -4 \\
0 & 1/2 & -2 & -1 & -5/2 & -4 \\
0 & 1/2 & 2 & 2 & 9/2 & 8 \\
0 & -1/3 & -8/3 & -2/3 & -3 & -16/3 \\
0 & 1/2 & 4 & 2 & 15/2 & 16 \\
0 & -1/6 & -4/3 & -4/3 & -9/2 & -32/3 \\
0 & -1/4 & -4 & -1/2 & -17/4 & -8 \\
0 & 11/24 & 22/3 & 11/6 & 97/8 & 88/3 \\
0 & -1/4 & -4 & -2 & -45/4 & -32 \\
0 & 1/24 & 2/3 & 2/3 & 27/8 & 32/3 \\
0 & -1/5 & -32/5 & -2/5 & -33/5 & -64/5 \\
0 & 5/12 & 40/3 & 5/3 & 81/4 & 160/3 \\
0 & -7/24 & -28/3 & -7/3 & -183/8 & -224/3 \\
0 & 1/12 & 8/3 & 4/3 & 45/4 & 128/3 \\
0 & -1/120 & -4/15 & -4/15 & -81/40 & -128/15
\end{pmatrix}$$